Lecture 2 Simple models of astrophysical fluids and their motions

In the previous lecture we established the momentum equation (1.9), the continuity equation (1.5), Poisson's equation(1.13) and the energy equation (1.17). Assuming that the only body forces are due to self-gravity, so that $\mathbf{f} = -\nabla\psi$ in equation (1.9), these equations are:

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p - \rho\nabla\psi\,, \qquad (2.1)$$

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{u} = 0, \qquad (2.2)$$

$$\nabla^2\psi = 4\pi G\rho\;, \qquad (2.3)$$

$$\frac{DU}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} = \varepsilon - \frac{1}{\rho}\nabla\cdot\mathbf{F}. \qquad (2.4)$$

Note that these contain seven dependent variables, namely ρ, the three components of $\mathbf{u}$, p, ψ and U. The three components of (2.1), together with (2.2) - (2.4), provide six equations, and a seventh is the equation of state (*e.g.* that for an ideal gas) which provides a relation between any three thermodynamic state variables, so that (for example) the internal energy U and temperature T can be written in terms of p and ρ. (ε and $\mathbf{F}$ are assumed to be known functions of the other variables). Thus one might hope in principle to solve these equations, given suitable boundary conditions. In practice this set of equations is intractable to exact solution, and one must resort to numerical solutions. Even these can be extremely problematic so that, for example, understanding turbulent flows is still a very challenging research area. Moreover, an analytic solution to a somewhat idealized problem may teach one much more than a numerical solution. One useful idealization is where we assume that the fluid velocity and all time derivatives are zero. These are called equilibrium solutions and describe a steady state. Although a true steady state may be rare in reality, the time-scale over which an astrophysical system evolves may be very long, so that at any particular time the state of many astrophysical fluid bodies may be well represented by an equilibrium model. Even when the dynamical behaviour of the body is important, it can often be described in terms of small departures from an equilibrium state. Hence in this lecture we start by looking at some equilibrium models and then derive equations describing small perturbations about an equilibrium state.

2.1 Hydrostatic equilibrium for a self-gravitating body

If we suppose that $\mathbf{u} = 0$ everywhere, and that all quantities are independent of time, then equation (2.1) becomes

$$\nabla p + \rho \nabla \psi = 0 ; \qquad (2.5)$$

the continuity equation becomes trivial; and equation (2.3) is unchanged. A fluid satisfying equation (2.5) is said to be in hydrostatic equilibrium. If it is self-gravitating (so that ψ is determined by the density distribution within the fluid), then equation (2.3) must also be satisfied.

Putting $\mathbf{u} = 0$ and $\partial / \partial t = 0$ in equation (2.4), we obtain that the heat sources given by ε must be exactly balanced by the heat flux term $\rho^{-1} \nabla \cdot \mathbf{F}$. If this holds, then the fluid is also said to be in thermal equilibrium. Since we have not yet considered what the heat sources might be, nor the details of the heat flux, we shall neglect considerations of thermal equilibrium at this point. Further reading material on the topics of this section may be found in [2.1], [2.2], [2.4] and [2.5].

2.1.1 Spherically symmetric case

Mass inside a sphere of radius r, centred on the origin, is

$$m(r) = \int_0^r 4\pi r^2 \rho(r)\, dr. \qquad (2.6)$$

Gravitational potential ψ is only a function of r; integrating the Poisson's equation (2.3) over the spherical volume gives

$$\nabla \psi = \frac{Gm}{r^2} \hat{\mathbf{r}} , \qquad (2.7)$$

$\hat{\mathbf{r}}$ being a unit vector in the radial direction - a result which could well be anticipated. (NB $\nabla \psi$ is minus the gravitational acceleration $\mathbf{g}$.) Also, by equation (2.5),

$$\nabla p = -\frac{Gm\rho}{r^2} \hat{\mathbf{r}} . \qquad (2.8)$$

The vector ∇p points towards the origin, so the pressure decreases as r increases.

One can only make further progress by assuming some relation between pressure and density. Suppose then that the fluid is an ideal gas, so

$$p = \frac{NkT}{V} = \frac{\Re \rho T}{\mu} \equiv a^2 \rho, \qquad (2.9)$$

where $\Re$ is the universal gas constant and μ is the molecular weight. a is known as the isothermal sound speed. Suppose further that the temperature T, as well as μ, are both constants throughout the fluid, so a is also a constant. Then equation (2.8) becomes

$$a^2 \frac{d\rho}{dr} = -\frac{Gm\rho}{r^2} \quad ,$$

which implies that

$$\frac{d}{dr}\left(\frac{r^2 a^2}{\rho}\frac{d\rho}{dr}\right) = -4\pi G r^2 \rho. \qquad (2.10)$$

Seeking a solution of the form $\rho = Ar^n$, where A and n are constants, gives

$$\rho = \frac{a^2}{2\pi G r^2}, \qquad p = \frac{a^4}{2\pi G r^2}. \qquad (2.11)$$

This is the singular self-gravitating isothermal sphere solution. It is not physically realistic at $r = 0$, where p and ρ are singular, but nonetheless it is a useful analytical model solution. Of course, in a real nondegenerate star, for example, the interior is not isothermal: the temperature increases with depth, which in turn means that the pressure increases and the star is prevented from collapsing in upon itself without recourse to infinite pressure and density at the centre.

Exercise 2.1. Verify that (2.11) is a solution of (2.10). Verify also that $a^2 = p / \rho$ has dimensions of velocity squared. 

2.1.2 Plane-parallel layer under constant gravity

In modelling the atmosphere and outer layers of a star, the spherical geometry can often be ignored, so that such a region can be approximated as a plane-parallel layer. Moreover, in the rarified outer layers of a star the gravitational acceleration g may be approximated as a constant vector. Thus, in Cartesian coordinates x, y, z we have a region in which everything is a function of z alone and (taking $\hat{\mathbf{z}}$ pointing downwards), $\mathbf{g} = g\hat{\mathbf{z}}$, where g is constant. Hence (2.5) becomes

$$\frac{dp}{dz} = g\rho(z). \qquad (2.12)$$

Since self-gravity is being ignored, equation (2.3) is not used.

In the *isothermal* case ($p/\rho = a^2$ constant), equation (2.12) can be integrated to give

$$\rho = \rho_0 \exp(gz/a^2) \qquad (2.13)$$

where the constant ρ_0 is the density at $z = 0$. The density scale height H is defined by

$$H = \left| \frac{1}{\rho} \frac{d\rho}{dz} \right|^{-1}. \qquad (2.14)$$

Hence, in this case, $H = a^2/g$ and is constant. Thus $\rho = \rho_0 \exp(z/H)$.

Exercise 2.2. Derive the solution (2.13). 

2.2 Small perturbations about equilibrium

In many interesting instances, such as the oscillations of a Cepheid, the motion of a fluid body can be considered to be small disturbances about an equilibrium state. Suppose that in equilibrium the pressure, density and gravitational potential are given by $p = p_0$, $\rho = \rho_0$, $\psi = \psi_0$ (all possibly functions of position, but independent of time) and of course $\mathbf{u} = 0$. Using equations (2.5) and (2.3), the equilibrium quantities satisfy

$$\nabla p_0 = -\rho_0 \nabla \psi_0 ,$$
$$\nabla^2 \psi_0 = 4\pi G \rho_0 . \qquad (2.15)$$

Suppose now that the system undergoes small motions about the equilibrium state, so

$$p = p_0 + p', \quad \rho = \rho_0 + \rho', \quad \psi = \psi_0 + \psi', \quad (2.16)$$

so for example $p'(\mathbf{r},t) \equiv p(\mathbf{r},t) - p_0(\mathbf{r},t)$ is the difference between the actual pressure and its equilibrium value at position $\mathbf{r}$. Substituting these expressions into equations (2.1) - (2.3) yields

$$\begin{aligned}
&(\rho_0 + \rho')\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) \\
&\quad = -\nabla(p_0 + p') - (\rho_0 + \rho')\nabla(\psi_0 + \psi'), \\
&\frac{\partial}{\partial t}(\rho_0 + \rho') = -\nabla\cdot\left((\rho_0 + \rho')\mathbf{u}\right), \qquad (2.17) \\
&\nabla^2(\psi_0 + \psi') = 4\pi G(\rho_0 + \rho').
\end{aligned}$$

We suppose that the perturbations (the primed quantities and the velocity) are *small*; hence we neglect the products of two or more small quantities, since these will be even smaller. This is known as linearizing, because we only retain equilibrium terms and terms that are linear in small quantities. This simplifies equations (2.17) to:

$$\begin{aligned}
&\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -\nabla(p_0 + p') - (\rho_0 + \rho')\nabla\psi_0 - \rho_0\nabla\psi', \\
&\frac{\partial \rho'}{\partial t} = -\nabla\cdot(\rho_0\mathbf{u}), \qquad (2.18) \\
&\nabla^2(\psi_0 + \psi') = 4\pi G(\rho_0 + \rho').
\end{aligned}$$

Subtracting equations (2.15) leaves a set of equations all the terms of which are linear in small quantities (*e.g.* [2.3]):

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -\nabla p' - \rho'\nabla\psi_0 - \rho_0\nabla\psi', \qquad (2.19a)$$

$$\frac{\partial \rho'}{\partial t} = -\nabla\cdot(\rho_0\mathbf{u}), \qquad (2.19b)$$

$$\nabla^2\psi' = 4\pi G\rho'. \qquad (2.19c)$$

Equations (2.19) give 5 equations (counting the vector equation as three) for 6 unknowns ($\mathbf{u}, p', \rho', \psi'$). We need another equation to close the system: that equation comes from energy considerations.

In full generality, we should perturb the energy equation (2.4) in the same manner as equations (2.1) - (2.3). But there are two limiting cases which are sufficiently common to be very useful and are simpler than using the full perturbed equation (2.4) because they don't involve a detailed description of how ε and $\mathbf{F}$ are perturbed.

Adiabatic fluctuations

Let the typical time scale and length scale on which the perturbations vary be T and λ, respectively. If T is much shorter than the timescale on which heat can be exchanged over a distance λ, then we can say that over a timescale T the heat gained or lost by a fluid element is zero: $\delta Q = 0$. In the previos lecture we established the adiabatic relation between the material derivatives if pressure and density (equation 1.22):

$$\frac{Dp}{Dt} = \frac{\Gamma_1 p}{\rho}\frac{D\rho}{Dt}. \qquad (2.20)$$

The linearized form of this equation is

$$\frac{\partial p'}{\partial t} + \mathbf{u}\cdot\nabla p_0 = \frac{\Gamma_1 p_0}{\rho_0}\left(\frac{\partial \rho'}{\partial t} + \mathbf{u}\cdot\nabla \rho_0\right). \qquad (2.21)$$

(In the last equation, the adiabatic exponent Γ_1 is also an equilibrium quantity because we have linearized, but for clarity the zero subscript has been omitted).

Isothermal fluctuations

The converse situation is where the timescale for heat exchange between neighbouring material is much shorter than the timescale of the perturbations. Since heat tends to flow from hotter regions to cooler ones, efficient heat exchange will eliminate any temperature fluctuations. Assuming an ideal gas, perturbing equation (2.9) gives

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T}. \qquad (2.22)$$

For isothermal fluctuations, $dT = 0$. Hence $dp/p = d\rho/\rho$. In terms of material derivatives,

$$\frac{Dp}{Dt} = \frac{p}{\rho}\frac{D\rho}{Dt}. \qquad (2.23)$$

Linearizing this gives an equation of the same form as equation (2.21) but without the factor Γ_1.

2.3 Lagrangian perturbations

We have previously considered perturbations evaluated at a fixed point in space, so for example $p'(\mathbf{r},t) \equiv p(\mathbf{r},t) - p_0(\mathbf{r},t)$ is the difference between the actual pressure and the value it would take in equilibrium *at that same point in space*. One can also evaluate perturbations as seen by a fluid element (cf. the material derivative). Such a perturbation will be denoted δp, for example. Now $\delta\mathbf{r}$ is the displacement of a fluid element from the position it would have been at in equilibrium.

$$\begin{aligned}\delta p &\equiv p(\mathbf{r}_0 + \delta\mathbf{r}) - p_0(\mathbf{r}_0) \\ &= p(\mathbf{r}_0) + \delta\mathbf{r}\cdot\nabla p_0 - p_0(\mathbf{r}_0),\end{aligned} \qquad (2.24)$$

where $\mathbf{r}_0$ is the equilibrium position of the fluid element; in the second equation, the first two terms of a Taylor expansion of $p(\mathbf{r}_0 + \delta\mathbf{r})$ have been taken: strictly we should have $\delta\mathbf{r}\cdot\nabla p$, but $\delta\mathbf{r}\cdot\nabla p_0$ is correct up to terms linear in small quantities. Equation (2.24) can be written

$$\delta p(\mathbf{r}_0) = p'(\mathbf{r}_0) + \delta\mathbf{r}\cdot\nabla p_0 \qquad (2.25)$$

where the argument on the left is written $\mathbf{r}_0$ (rather than $\mathbf{r}$) and this is again correct in linear theory. Of course, equation (2.25) holds for any quantity, not just pressure. We note that, in linear theory, $\partial\,\delta f/\partial t = D\,\delta f/Dt$ (where f is any quantity); hence since the velocity of the fluid is just the rate of change of position as seen by a fluid element,

$$\mathbf{u} = \frac{D\,\delta\mathbf{r}}{Dt} = \frac{\partial\,\delta\mathbf{r}}{\partial t}. \qquad (2.26)$$

Perturbations p' at a fixed point in space are called Eulerian; perturbations δp following the fluid are called Lagrangian. See for example [2.1].

2.4 Sound waves

The linearized perturbed Poisson equation (2.19c) has formal solution

$$\psi'(\mathbf{r}) = \int \frac{-G\rho'(\tilde{\mathbf{r}})}{|\mathbf{r}-\tilde{\mathbf{r}}|} d\tilde{V}, \qquad (2.27)$$

the integration being over the whole volume of the fluid. What is under the itegral, is just the perturbation to ψ induced by the mass perturbation in volume $d\tilde{V}$. In the integral on the right-hand side of (2.27) the positive and negative fluctuations in ρ' tend to cancel out, so that it is often a reasonable approximation to say that $\psi' \approx 0$. Thus we will frequently drop ψ' in equation (2.19a). We shall do so in the remainder of this lecture, for example. The term is also absent in equation (2.19a) in problems where self-gravitation is ignored altogether. The term is very important, however, in Lecture 3 when we discuss the Jeans instability.

Suppose now that we have a *homogeneous* medium, so that equilibrium quantities are independent of position (and hence in particular $\nabla p_0 = 0 = \nabla\psi_0$). Equations (2.19a) and (2.19b) can then be rewritten

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -\nabla p', \qquad \frac{\partial \rho'}{\partial t} = -\rho_0 \nabla \cdot \mathbf{u}, \qquad (2.28)$$

so taking the divergence of the first of these equations and substituting for $\nabla \cdot \mathbf{u}$ from the second gives

$$\frac{\partial^2 \rho'}{\partial t^2} = \nabla^2 p'. \qquad (2.29)$$

Suppose further that the perturbations are adiabatic. Now equation (2.21) for a homogeneous medium becomes

$$\frac{\partial p'}{\partial t} = c^2 \frac{\partial \rho'}{\partial t},$$

where $c^2 \equiv \Gamma_1 p_0 / \rho_0$ is a constant. Integrating with respect to time gives

$$p' = c^2 \rho',$$

which can be used to eliminate p' from equation (2.29):

$$\frac{\partial^2 \rho'}{\partial t^2} = c^2 \nabla^2 \rho'. \qquad (2.30)$$

This is a wave equation (cf. the 1-D analogue $\partial^2 y / \partial t^2 = c^2 \partial^2 y / \partial x^2$) and describes sound waves propagating with speed c (see [2.2]). In fact, c is called the adiabatic sound speed. (If we had instead assumed isothermal fluctuations, we would have obtained a wave equation with c replaced by a, the isothermal sound speed; cf. section 2.1.1).

One can seek plane wave solutions of eq. (2.30):

$$\rho' = A \exp\left(i\mathbf{k} \cdot \mathbf{r} - i\omega t\right), \qquad (2.31)$$

where the amplitude A, frequency ω and wavenumber $\mathbf{k}$ are constants. (Here and elsewhere, it is understood when writing complex quantities that the real part should be taken to get a physically meaningful solution.) Substituting equation (2.31) into (2.30), one finds that ρ' is a nontrivial solution ($A \neq 0$) provided

$$\omega^2 = c^2 \left|\mathbf{k}\right|^2. \qquad (2.32)$$

This is known as the *dispersion relation* for the waves. It specifies the relation that must hold between the freqency and wavenumber for the wave to be a solution of the given wave equation. With a suitable choice of phase, one can deduce from (2.31) that

$$\begin{aligned} \rho' &= A \cos\left(\mathbf{k} \cdot \mathbf{r} - \omega t\right), \\ p' &= A c^2 \cos\left(\mathbf{k} \cdot \mathbf{r} - \omega t\right), \qquad (2.33) \\ \delta \mathbf{r} &= -\frac{c^2}{\rho_0 \omega^2} A \mathbf{k} \sin\left(\mathbf{k} \cdot \mathbf{r} - \omega t\right). \end{aligned}$$

Note that the adiabatic pressure and density fluctuations are in phase, while the displacement is $\pi / 2$ out of phase. A sound wave is called longitudinal, because the fluid displacement is parallel to the wavenumber $\mathbf{k}$.

Exercise 2.3. Derive the expressions (2.33). Explain physically the phase difference between the displacement and the other two quantities in a plane sound wave.

2.5 Surface gravity waves

As a second example of a simple wave solution of the linearized perturbed fluid equations, consider an incompressible fluid ($\nabla \cdot \mathbf{u} = 0$) of constant density ρ_0, occupying the region $z \geq 0$ below the free surface $z = 0$ (so p is constant at the surface). Suppose also that gravity $g\hat{z}$ is uniform and points downwards, and that self-gravity is negligible. This is a reasonable model for ocean waves on deep water, for example. Equation (2.19b) implies that $\rho' = 0$. Hence equation (2.19a) becomes

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -\nabla p', \qquad (2.34)$$

and taking the divergence of this gives

$$\nabla^2 p' = 0. \qquad (2.35)$$

We seek a solution with sinusoidal horizontal variation in the x direction:

$$p'(x,z,t) = f(z)\cos(kx - \omega t), \qquad (2.36)$$

where f is an as yet unknown function; and without loss of generality $k > 0$. Substituting this into (2.35) gives

$$\frac{d^2 f}{dz^2} = k^2 f$$

whence

$$f(z) = A\exp(-kz) + B\exp(kz) \qquad (2.37)$$

(see [2.2]). The fluid is infinitely deep, and the solution should not become infinite as $z \to \infty$; hence $B = 0$.

The boundary condition at the free surface $z = 0$ is that the pressure at the edge of the fluid should be constant: hence $\delta p = 0$ at $z = 0$. Thus at $z = 0$

$$\delta p = p' + \delta \mathbf{r} \cdot \nabla p_0 = p' + \rho_0 g \hat{\mathbf{z}} \cdot \delta \mathbf{r} = 0. \qquad (2.38)$$

Taking the dot product of (2.34) with $\hat{\mathbf{z}}$ and using (2.36) and (2.37) yields

$$\hat{\mathbf{z}} \cdot \delta \mathbf{r} = -\frac{k}{\rho_0 \omega^2} p' \qquad (2.39)$$

everywhere. Hence the boundary condition (2.38) can only be satisfied if ω and k satisfy the dispersion relation

$$\omega^2 = gk. \qquad (2.40)$$

It is clear that these are surface waves; for the perturbed quantities all decrease exponentially with depth. In reality, of course, the fluid cannot be infinitely deep, so B is not identically zero. Instead, A and B will have to be chosen such that some boundary condition is satisfied at the bottom of the fluid layer. However, provided the depth of the layer is much greater than k^{-1}, it will generally be the case that B has to be much less than A.

LITERATURE

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