Lecture 1 Basic fluid equations

The material in sections 1.1, 1.2 and 1.3 can be found in many texts on hydrodynamics or fluid mechanics. Good treatments can be found in references [1.1] and [1.2]. A different approach is followed by [1.3].

1.1 The material derivative

$\partial / \partial t$ will denote the rate of change of some physical quantity with respect to time *at a fixed position in space*.

D / Dt (the **material derivative**) will denote the rate of change of some quantity with respect to time but travelling along with the fluid.

Let f be any quantity (e.g. temperature). Then

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f, \qquad (1.1)$$

where $\mathbf{u}(\mathbf{r},t)$ is the velocity of the fluid at position $\mathbf{r}$ and time t.

Exercise 1.1 Convince yourself that if the temperature of the fluid varies with position, but that the temperature of any particular parcel of fluid does not change with time, then the rate of change of temperature with time as seen by an observer at a fixed point in space is $-\mathbf{u} \cdot \nabla f$, in agreement with equation (1.1) when $Df / Dt = 0$. Conversely, convince yourself also that equation (1.1) gives the correct result for the change as seen by the fluid when $\partial f / \partial t = 0$. If you need help, press .

1.2 The continuity equation

Consider a volume V, which is fixed in space. The total mass of fluid in V is $\int_V \rho \, dV$, where $\rho(\mathbf{r},t)$ is the density of the fluid. The time derivative of the mass in V is the mass flux into V across its surface S, *i.e.*

$$\frac{d}{dt} \int_V \rho \, dV = -\int_S (\rho \mathbf{u}) \cdot \mathbf{n} \, dS \;, \qquad (1.2)$$

where $\mathbf{n}$ is the outward normal to the surface S. Hence, using the divergence theorem, we obtain

$$\int_V \frac{\partial \rho}{\partial t}\, dV = -\int_S \rho \mathbf{u}\cdot\mathbf{n}\, dS = -\int_V \nabla\cdot(\rho\mathbf{u})\, dV. \qquad (1.3)$$

Since this holds for arbitrary V, it follows that

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0. \qquad (1.4)$$

This is the continuity (or mass conservation) equation. Using equation (1.1) this can also be written as

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{u} = 0. \qquad (1.5)$$

1.3 The momentum equation

One can similarly derive a momentum equation, or equation of motion, for the fluid by considering the rate of change of the total momentum of the fluid inside a volume V. It turns out to be easiest to consider a volume *moving with the fluid*, so that no fluid is flowing across its surface into or out of V. The momentum of the fluid in V is $\int_V \rho\mathbf{u}\, dV$, and the rate of change of this momentum is equal to the net force acting on the fluid in volume V. These are of two kinds. First there are body forces, such as gravity, which act on the particles inside V: their net effect is a force

$$\int_V \rho\mathbf{f}\, dV,$$

where $\mathbf{f}$ is the body force per unit mass. (Nb force per unit mass has dimensions of acceleration.) E.g., $\mathbf{f}$ could be the gravitational acceleration $\mathbf{g}$. The second kind of forces acting are surface forces - forces exerted on the surface S of V by the surrounding fluid. In an *inviscid* fluid, such as we mostly generally be considering, the surface force acts normally to the surface and its net effect is

$$-\int_S p\mathbf{n}\, dS,$$

p being the pressure. Equating force to change of momentum we obtain

$$\frac{d}{dt}\int_V \rho \mathbf{u}\, dV = -\int_S p\mathbf{n}\, dS + \int_V \rho \mathbf{f}\, dV. \qquad (1.6)$$

Since $\rho\, dV$, the mass of a fluid element, is invariant following the motion,

$$\frac{d}{dt}\int_V \rho \mathbf{u}\, dV = \int_V \rho \frac{D\mathbf{u}}{Dt} dV \qquad (1.7)$$

and hence, applying the divergence theorem to the surface integral in equation (1.6), we obtain

$$\int_V \rho \frac{D\mathbf{u}}{Dt} dV = \int_V (-\nabla p + \rho \mathbf{f})\, dV. \qquad (1.8)$$

Since this holds for arbitary material volume V, it follows that

$$\rho \frac{D\mathbf{u}}{Dt} \equiv \rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \rho \mathbf{f}. \qquad (1.9)$$

This is the momentum equation for an inviscid fluid.

Taking into account the viscous forces would add the right-hand side of the momentum equation with an additional term $\mu\left(\nabla^2\mathbf{u} + \frac{1}{3}\nabla(\nabla\cdot\mathbf{u})\right)$, where μ is dynamic viscosity - a result which we quote here without derivation. Note that the viscous forces can play a key role in some astrophysical applications which we will not consider in these lectures (in particular, accretion discs).

1.4 Newtonian gravity

A mass m' at position $\mathbf{r}'$ exerts on any other mass m at position $\mathbf{r}$ an attractive force $\mathbf{F} = m\mathbf{g}(\mathbf{r})$; the gravitational acceleration $\mathbf{g}(\mathbf{r})$ can be written as the gradient of a potential function, $\mathbf{g} = -\nabla\psi$, where

$$\psi = -\frac{Gm'}{|\mathbf{r}-\mathbf{r}'|}. \qquad (1.10)$$

Let S be a spherical surface of radius $|\mathbf{r}-\mathbf{r}'|$ centered at $\mathbf{r}'$. With $|\nabla\psi| = Gm'/|\mathbf{r}-\mathbf{r}'|^2$, we have

$$\int_S \mathbf{n}\cdot\nabla\psi\, dS = 4\pi Gm', \qquad (1.11)$$

the result which does not depend on $|\mathbf{r}-\mathbf{r}'|$. It can also be verifyed directly that the gravitational potential of our point mass (equation 1.10) satisfyes $\nabla\cdot\nabla\psi \equiv \nabla^2\psi = 0$ (the Laplace equation) everywhere exept of just one point, $\mathbf{r}=\mathbf{r}'$. Using the divergence theorem, we observe that the surface S in the equation (1.11) can in fact be any (not necessary spherical) surface surrounding m'.

The gravitational field due to a fluid can be written as a potential, namely the sum of the potentials due to all the fluid elements. Summing over all the fluid elements inside S, and applying the divergence theorem once more, we get

$$\int_V \nabla^2\psi\, dV = 4\pi G\int_V \rho\, dV, \qquad (1.12)$$

where V is volume inside S. Since V is arbitrary, this equation can be rewritten as a partial differential equation, Poisson's equation:

$$\nabla^2\psi = 4\pi G\rho\ . \qquad (1.13)$$

1.5. The mechanical and thermal energy equations

If one takes Newton's third law, $F = ma = m(dv/dt)$ and multiplies by velocity v, one obtains that rate of work of the forces, Fv, is equal to the rate of change of kinetic energy, $d\left(\frac{1}{2}mv^2\right)/dt$.

Similarly, taking the dot product of the equation of motion for a fluid, (1.9), with the fluid velocity $\mathbf{u}$ yields

$$\frac{D}{Dt}\left(\frac{1}{2}\mathbf{u}^2\right) = -\frac{1}{\rho}\mathbf{u}\cdot\nabla p + \mathbf{u}\cdot\mathbf{f}\,. \qquad (1.14)$$

Equation (1.14) says that the rate of change of the kinetic energy of a unit mass of fluid is equal to the rate at which work is done on the fluid by pressure and body forces. This is sometimes called the mechanical energy equation.

Exercise 1.2. Prove equation (1.14).

An equation for the total energy - kinetic and internal thermal energy - can be derived in the same manner as was the momentum equation in Section 1.3. Let the internal energy per unit mass of fluid be U. Then the rate of change of kinetic plus internal energy of a material volume (*i.e.* one moving with the fluid) must be equal to the rate of work done on the fluid by surface and body forces, plus the rate at which heat is added to the fluid. Heat can be added in two ways: one is by its being generated at a rate ε per unit mass within the fluid volume (*e.g.* by nuclear reactions), while the second is by the flux of heat $\mathbf{F}$ into the volume from the surroundings (*e.g.* by radiation). Thus

$$\frac{d}{dt}\int_V \left(\frac{1}{2}\mathbf{u}^2 + U\right)\rho\, dV = \int_S \mathbf{u}\cdot(-p\mathbf{n})dS$$

$$+\int_V \mathbf{u}\cdot\mathbf{f}\rho\, dV + \int_V \varepsilon\rho\, dV + -\int_S \mathbf{F}\cdot\mathbf{n}\, dS. \qquad (1.15)$$

In the same way as for the momentum equation, one rewrites all the surface integrals in this equation as volume integrals, using the divergence theorem. The resulting equation holds for an arbitrary volume V and so one deduces that

$$\rho\left(\frac{D}{Dt}\left(\tfrac{1}{2}\mathbf{u}^2\right) + \frac{DU}{Dt}\right)$$

$$= -\nabla\cdot(p\mathbf{u}) + \rho\mathbf{u}\cdot\mathbf{f} + \rho\varepsilon - \nabla\cdot\mathbf{F}. \qquad (1.16)$$

One can derive an equation for the thermal energy alone by dividing (1.16) by the density and then subtracting the kinetic energy equation (1.14):

$$\frac{DU}{Dt} = \frac{p}{\rho^2}\frac{D\rho}{Dt} + \varepsilon - \frac{1}{\rho}\nabla\cdot\mathbf{F}. \qquad (1.17)$$

Note that the divergence of $\mathbf{u}$ has been replaced by $-\rho^{-1}D\rho/Dt$ using the continuity equation (1.5).

Noting that the volume per unit mass is just the reciprocal of the density, *i.e.* $V = \rho^{-1}$, we recognise the thermal energy equation (1.17) as a statement of the first law of thermodynamics:

$$dU = -p\,dV + \delta Q\,, \qquad (1.18)$$

that is, the change in the internal energy is equal to the work $-p\,dV$ done (on the fluid) plus the heat added. Note that p, V, U are properties of the fluid (in fact they are thermodynamic state variables) and we denote changes in them with the symbol d. In contrast, there is no such property as the heat content and so we cannot speak of the change of heat content. Instead, we can only speak of the heat added, and we therefore use a different notation, *i.e.* δQ. The second law of thermodynamics states that

$$\delta Q = T\,dS\,, \qquad (1.19)$$

where S is a thermodynamic state variable, the *specific entropy* (*i.e.* the entropy per unit mass). Combining this with the first law, equation (1.18), yields

$$dU = T\,dS - p\,dV\,. \qquad (1.20)$$

1.6. Adiabatic approximation. Ideal gases

In practical applications, the thermal energy equation (1.17) can often be simplified. The most important simplification comes from the so-called adiabatic approximation. In the adiabatic approximation, we neglect any heat generation inside the fluid element and any heat exchange with the surroundings, which means setting $\varepsilon = \mathbf{F} = \delta Q = dS = 0$ (the entropy is conserved).

For the adiabatic changes, pressure and density variations in a fluid element are related with each other through the so-called adiabatic exponent

$$\Gamma_1 = \left(\frac{\partial \ln p}{\partial \ln \rho}\right)_S\,, \qquad (1.21)$$

subscript S designates that the partial derivative is taken at constant entropy. In terms of material derivatives, the thermal energy equation (1.17) is then equivalent to

$$\frac{Dp}{Dt} = \frac{\Gamma_1 p}{\rho}\frac{D\rho}{Dt}\,. \qquad (1.22)$$

To evaluate the adiabatic exponent Γ_1, we need to know the equation of state. The simplest (and often quite accurate) approximation here is the equation of state of an ideal gas,

$$pV = NkT, \tag{1.23}$$

where N is total number of particles (molecules or atoms) in volume V, and k is Boltzmann constant. In the ideal gas, interactions between particles are neglected, and the internal inergy is just the sum of the kinetic energies of all the free particles.

For a single particle, the kinetic energy is $kT/2$ times the number of degrees of freedom in its free motion. For a monoatomic gas (in which the particles can be considered as point masses) there are three degrees of freedom (three possible directions of translational motion), and we have

$$U = \frac{3}{2}NkT. \tag{1.24}$$

This simple expression for the internal energy gives the adiabatic exponent of an ideal monoatomic gas as

$$\Gamma_1 = \frac{5}{3}. \tag{1.25}$$

Exercise 1.3. Fill in the missing steps to derive equation (1.25).

LITERATURE

[1.1] Batchelor, G. K., 1967. *An Introduction to Fluid Dynamics* (C.U.P.: Cambridge)

[1.2] Landau, L. D., Lifshitz, E. M., 1959. *Fluid Mechanics* (Pergamon Press: Oxford)

[1.3] Shu, F. H., 1992. *The Physics of Astrophysics Vol. II. Gas Dynamics* (University Science Books: Mill Valley, CA)