EXERCISE 6.1

With $\partial \mathbf{u} / \partial t = \partial^2 \delta \mathbf{r} / \partial t^2 = -\omega^2 \delta \mathbf{r}$, and

$\nabla p' = \hat{\mathbf{r}} \frac{\partial}{\partial r} p' + \frac{1}{r} \nabla_1 p'$, and $\nabla \psi_0 = \hat{\mathbf{r}} g_0$, the radial component of the momentum equation (6.7a) reduces to (6.9a), and its horizontal component - to (6.9b).

With $\nabla \cdot (\rho_0 \mathbf{u}) = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \rho_0 u_r \right) + \frac{1}{r} \nabla_1 \cdot (\rho_0 \mathbf{u})$ and

$$\nabla_1 \cdot (\rho_0 \mathbf{u}) = \rho_0 \nabla_1 \cdot \mathbf{u} = i\omega \rho_0 \nabla_1 \cdot \delta \mathbf{r}$$

$$= i\omega \rho_0 V \nabla_1^2 Y_{\ell m} = -i\omega \ell (\ell + 1) \rho_0 V Y_{\ell m},$$

the continuity equation (6.7b) gives (6.9c). With $\nabla p_0 = \hat{\mathbf{r}} \frac{dp_0}{dr}$ and

$\nabla \rho_0 = \hat{\mathbf{r}} \frac{d\rho_0}{dr}$, the adiabatic energy equation (6.7c) gives (6.9d).

Now express ρ_1 in terms of p_1 and U from the equation (6.9d):

$$\rho_1 = \frac{1}{c^2} p_1 - \left(\frac{d\rho_0}{dr} - \frac{1}{c^2} \frac{dp_0}{dr} \right) U = \frac{1}{c^2} p_1 + \frac{\rho_0}{g_0} N^2 U$$

and $V = p_1 / (\rho_0 \omega^2)$ from the equation (6.9b), and substitute into (6.9a,c) to eliminate ρ_1 and V and to get (6.10b,a).

EXERCISE 6.2

We need to prove that the equation

$$F(\tilde{w}) = \int_{r_1}^{R} \left(\frac{r^2}{c^2} - \tilde{w}^2 \right)^{1/2} \frac{dr}{r}, \qquad (6.20)$$

when considered as an integral equation with function $F(\tilde{w})$ specified, has a solution

$$\ln\frac{r_1}{R} = \frac{2}{\pi}\int_{r_1/c_1}^{R/c_s}\left(\tilde{w}^2 - \frac{r_1^2}{c_1^2}\right)^{-1/2}\frac{dF}{d\tilde{w}}\,d\tilde{w}\,, \qquad (6.22)$$

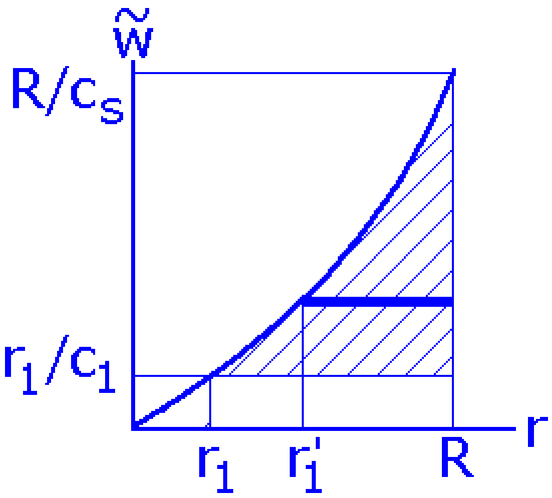
which allows to infer r_1 as function of r_1/c_1, and hence c as function of r.

Take the derivative of the both sides of (6.20) with respect to $\tilde{w}$, considered as a parameter in the integral:

$$\frac{dF(\tilde{w})}{d\tilde{w}} = -\tilde{w}\int_{r_1}^{R}\left(\frac{r^2}{c^2} - \tilde{w}^2\right)^{-1/2}\frac{dr}{r}\,,$$

and substitute the result into (6.22):

$$\ln\frac{r_1}{R} = -\frac{2}{\pi}\int_{\frac{r_1}{c_1}}^{\frac{R}{c_s}}\left(\tilde{w}^2 - \frac{r_1^2}{c_1^2}\right)^{-\frac{1}{2}} d\tilde{w}\,\tilde{w}\int_{r_1'}^{R}\left(\frac{r^2}{c^2} - \tilde{w}^2\right)^{-\frac{1}{2}}\frac{dr}{r}\,,$$



where r_1' is such that

$$r_1'/c(r_1') = \tilde{w}\,.$$

Now change the order of integration, using

$$2\tilde{w}\,d\tilde{w} = d(\tilde{w}^2)\,:$$

$$\ln\frac{r_1}{R} = -\frac{1}{\pi}\int_{r_1}^{R}\frac{dr}{r}\int_{\frac{r_1^2}{c_1^2}}^{\frac{r^2}{c^2}}\left(\tilde{w}^2-\frac{r_1^2}{c_1^2}\right)^{-\frac{1}{2}}\left(\frac{r^2}{c^2}-\tilde{w}^2\right)^{-\frac{1}{2}}d(\tilde{w}^2).$$

To evaluate the inner integral, simplify it by using new variable t (just linear rescaling of $\tilde{w}^2$) as

$$t = 2\,\frac{\tilde{w}^2 - r_1^2/c_1^2}{r^2/c^2 - r_1^2/c_1^2} - 1,$$

so that

$$\tilde{w}^2 - \frac{r_1^2}{c_1^2} = \frac{r^2/c^2 - r_1^2/c_1^2}{2}(1+t),$$

$$\frac{r^2}{c^2} - \tilde{w}^2 = \frac{r^2/c^2 - r_1^2/c_1^2}{2}(1-t),$$

$$d(\tilde{w}^2) = \frac{r^2/c^2 - r_1^2/c_1^2}{2}\,dt.$$

We thus have

$$\ln\frac{r_1}{R} = -\frac{1}{\pi}\int_{r_1}^{R}\frac{dr}{r}\int_{-1}^{1}\frac{dt}{(1-t^2)^{1/2}}.$$

With substitution $t = \sin(\theta)$, the inner integral is

$$\int_{-1}^{1}\frac{dt}{(1-t^2)^{1/2}} = \int_{-\pi/2}^{\pi/2}\theta\,d\theta = \pi,$$

and we arrive to the identity

$$\ln \frac{r_1}{R} = -\int_{r_1}^{R} \frac{dr}{r}.$$