EXERCISE 2.1

When we substitute $\rho = Ar^n$ into the equation (2.10), we get

$$a^2 n = -4\pi G A r^{n+2}.$$

For this equation to be valid at all r, we need $n + 2 = 0$, or $n = -2$. From the same equation we then get $A = a^2/(2\pi G)$, and hence (2.11) as the solution for $\rho(r)$.

The dimension of pressure is $ML^{-1}T^{-2}$, where M is (the dimension of) mass, L is length, and T is time. Dimension of density is ML^{-3}. Their ratio is L^2T^{-2}, the dimension of the velocity squared.

EXERCISE 2.2

When $p = a^2\rho$ with a^2 constant, equation (2.12) is

$$\frac{d\rho}{dz} = \frac{g}{a^2}\rho$$

with general solution (A is arbitrary constant)

$$\rho(z) = A\exp\left(\frac{gz}{a^2}\right).$$

Constant A gives the value of density at $z = 0$, and hence we have (2.13) as the solution.

EXERCISE 2.3

The first of the equations (2.33) is just the real part of (2.31); the second follows immediately with using $p' = c^2\rho'$. To derive the third, we need to use the momentum equation (2.28)

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -\nabla p'.$$

With p' specified by (2.33) as

$$p' = Ac^2 \exp\left(i\mathbf{k} \cdot \mathbf{r} - i\omega t\right)$$

and with

$$\nabla\left(\mathbf{k} \cdot \mathbf{r}\right) = \hat{\mathbf{x}}\frac{\partial}{\partial x}(k_x x) + \hat{\mathbf{y}}\frac{\partial}{\partial y}(k_y y) + \hat{\mathbf{z}}\frac{\partial}{\partial z}(k_z z) = \mathbf{k},$$

we have

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -Ac^2 i\mathbf{k} \exp\left(i\mathbf{k} \cdot \mathbf{r} - i\omega t\right).$$

According to the momentum equation, the time dependence of $\mathbf{u}$ is determined by the same factor $\exp(-i\omega t)$, and hence

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -i\omega\rho_0\mathbf{u} = -i\omega\rho_0 \frac{\partial \delta\mathbf{r}}{\partial t} = -\omega^2\rho_0\delta\mathbf{r}.$$

By comparing the last two equations, we get $\delta\mathbf{r}$; the real part of the result is just the third of the equations (2.33).

At a given point in space, pressure and density fluctuations are in phase: an increase of pressure (p' positive) happens at the compression phase (ρ' positive), and p' is zero when ρ' is zero. Having $\rho' = 0$ means having no compression, which can only happen when there is no gradient in the displacement field, i.e. $\delta\mathbf{r}$ is either at maximum or at minimum. The result is the $\pi/4$ phase difference between $\delta\mathbf{r}$ and ρ'.