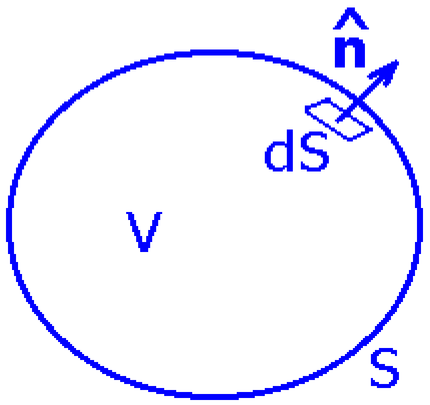
EXERCISE 1.1

Let temperature $T(\mathbf{r},t)$ varies in both space ($\nabla T \neq 0$) and time ($\partial T / \partial t \neq 0$), but appears to be constant in time for an observer moving together with a fluid element ($DT / Dt = 0$). In this situation, the contribution to the temperature variation in the element due to the variation of its position $\mathbf{r}$, this contribution being $\mathbf{u} \cdot \nabla T$, is compensated by the contribution due to the time dependence of $T(\mathbf{r},t)$, this second contribution being $\partial T / \partial t$. We have $\partial T / \partial t = -\mathbf{u} \cdot \nabla T$.

When $\partial T / \partial t = 0$, the temperature field $T(\mathbf{r},t)$ is stationary: at any point is space, T remains constant in time. When the fluid is in motion, the temperature of a fluid element can change in time due to its displacement to hotter or cooler regions. The rate of change will be $DT / Dt = \mathbf{u} \cdot \nabla T$.

THE DIVERGENCE THEOREM



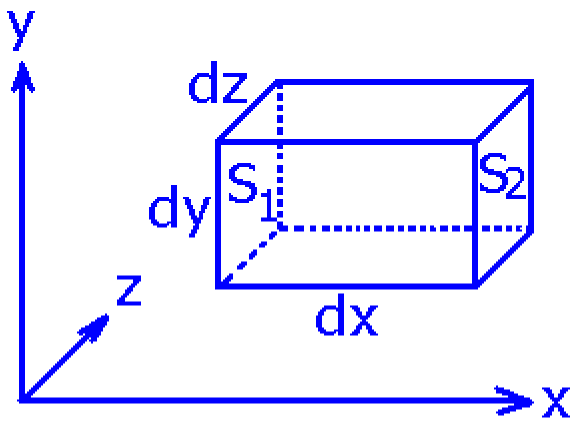
The divergence theorem (or Gauss theorem) states that for an arbitrary vector field $\mathbf{F}(\mathbf{r})$

$$\int_V \nabla \cdot \mathbf{F}\, dV = \int_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS,$$

for an arbitrary volume V bounded by a closed surface S. Here $\hat{\mathbf{n}}$ is unit vector normal to S and directed to the outside of volume V.

Proof. In cartesian coordinates x, y, z, consider a small rectangular element with dimensions dx, dy, dz. Let the element is small enough so that the divergence of the vector field $\mathbf{F} = (F_x, F_y, F_z)$, which is

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z},$$



can be considered uniform inside the element. The vector flux across the surface surrounding the element is the sum of six fluxes. Consider the contribution of the vector fluxes crossing the two surfaces S_1 and S_2, both orthogonal to x. This contribution is provided by a small increment in F_x due to the increment in x,

$$dF_x = \frac{\partial F_x}{\partial x} dx,$$

times the area of S_1 and S_2, which is $dy\,dz$. Adding similar contributions of the vector fluxes crossing the surfaces orthogonal to z and y, the total vector flux to the outside of the rectangular element is

$$\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) dx\,dy\,dz = \nabla \cdot \mathbf{F}\, dV.$$

Now any finite volume V can be considered as a sum of the small rectangular elements; summing $\nabla \cdot \mathbf{F}\, dV$, we get the volume integral in the divergence theorem. Summing the vector fluxes, the contributions coming from the rectangular areas separating any two neighbouring elements cancel out, and we are left with vector flux across the outer boundary S of V.

(Help 1-2)

Applition of the divergence theorem here might look unusual, because p is scalar and hence

$$\int_S p\hat{\mathbf{n}}\, dS$$

is vector, not a scalar. The procedure is nevertheless straightforward, if we consider the unit vector $\hat{\mathbf{n}}$ separately in tree components

$$\hat{\mathbf{n}} = (\hat{\mathbf{x}} \cdot \hat{\mathbf{n}})\hat{\mathbf{x}} + (\hat{\mathbf{y}} \cdot \hat{\mathbf{n}})\hat{\mathbf{y}} + (\hat{\mathbf{z}} \cdot \hat{\mathbf{n}})\hat{\mathbf{z}}$$

were hats denote unit vectors in cartesian coordinates (x, y, z). We then have

$$\int_S p\hat{\mathbf{n}}\, dS = \hat{\mathbf{x}}\int_S p\hat{\mathbf{x}}\cdot\hat{\mathbf{n}}\, dS + \hat{\mathbf{y}}\int_S p\hat{\mathbf{y}}\cdot\hat{\mathbf{n}}\, dS + \hat{\mathbf{z}}\int_S p\hat{\mathbf{z}}\cdot\hat{\mathbf{n}}\, dS .$$

We now apply the divergence theorem to the three integrals separately:

$$\int_S p\hat{\mathbf{n}}\, dS = \hat{\mathbf{x}}\int_V \nabla\cdot(p\hat{\mathbf{x}})\, dV + \hat{\mathbf{y}}\int_V \nabla\cdot(p\hat{\mathbf{y}})\, dV + \hat{\mathbf{z}}\int_V \nabla\cdot(p\hat{\mathbf{z}})\, dV ,$$

and with

$$\nabla\cdot(p\hat{\mathbf{x}}) = \partial p/\partial x, \nabla\cdot(p\hat{\mathbf{y}}) = \partial p/\partial y, \nabla\cdot(p\hat{\mathbf{z}}) = \partial p/\partial z ,$$

the result is

$$\int_S p\hat{\mathbf{n}}\, dS = \int_V \left(\frac{\partial p}{\partial x}\hat{\mathbf{x}} + \frac{\partial p}{\partial y}\hat{\mathbf{y}} + \frac{\partial p}{\partial z}\hat{\mathbf{z}}\right) dV = \int_V \nabla p\, dV .$$

Help 1-3

If you are not familiar with an expression for ∇^2 in spherical coordinates, you can work in cartesian coordinates (x, y, z), with

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} .$$

Choose the cartesian coordinate system with its origin $(0, 0, 0)$ at $\mathbf{r}'$ for convenience, so that

$$\psi = -\frac{Gm'}{(x^2 + y^2 + z^2)^{1/2}} .$$

You then have

$$\frac{\partial\psi}{\partial x} = Gm' \frac{x}{(x^2+y^2+z^2)^{3/2}}$$

and

$$\frac{\partial^2\psi}{\partial x^2} = Gm' \frac{x^2+y^2+z^2-3x^2}{(x^2+y^2+z^2)^{5/2}}.$$

Adding similar expressions for $\partial^2\psi/\partial y^2$ and $\partial^2\psi/\partial z^2$, you get $\nabla^2\psi = 0$ everywhere except of one point $(0,0,0)$.

EXERCISE 1.3

From the equation of state (1.23) and the epression (1.24) for the internal energy, we have

$$pV = \frac{2}{3}U.$$

For the adiabatic change, the first low of thermodynamics (1.18) is

$$dU = -pdV.$$

From these two equations,

$$pdV + Vdp = \frac{2}{3}dU = -\frac{2}{3}pdV,$$

or

$$\frac{dp}{p} = -\frac{5}{3}\frac{dV}{V}.$$

We can write $V = m/\rho$, and hence $dV/V = -d\rho/\rho$, whem mass m is small so that ρ can be considered uniform in V. We thus have

$$\frac{dp}{p} = \frac{5}{3}\frac{d\rho}{\rho},$$

i.e. $\Gamma_1 = 5/3$.