Lecture 6 Linear adiabatic nonradial oscillations. Helioseismology

6.1 Nonradial modes of oscillations of a star

In the last lecture we considered stellar oscillations where the motion was wholly in the radial direction. In this lecture we shall consider more general motions. The equilibrium structure about which the oscillations take place is still presumed to be spherically symmetric, but the velocity will now have a horizontal as well as a radial component, and the velocity and other perturbations will depend not only on r but also on θ and φ, where (r,θ,φ) are spherical polar coordinates.

A well-studied star in which nonradial oscillations are observed is the Sun. For this reason, we shall draw upon the example of the Sun frequently in this lecture. However, it should not be forgotten that other pulsating stars are known to exhibit nonradial oscillations (*e.g.* ZZ Cetis, Delta Scutis and Ap stars).

As a preliminary, we list some few mathematical expressions which will be used later in this Lecture. In spherical coordinates (r,θ,φ), the gradient, divergence and Laplacian operators can be written as

$$\nabla f = \hat{\mathbf{r}}\frac{\partial f}{\partial r} + \frac{1}{r}\nabla_1 f, \qquad (6.1)$$

$$\nabla_1 f = \hat{\theta}\frac{\partial f}{\partial \theta} + \frac{1}{\sin\theta}\hat{\varphi}\frac{\partial f}{\partial \varphi},$$

$$\nabla\cdot\mathbf{u} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r}\nabla_1\cdot\mathbf{u}, \qquad (6.2)$$

$$\nabla_1\cdot\mathbf{u} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, u_\theta\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}u_\varphi,$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\nabla_1^2 f, \qquad (6.3)$$

$$\nabla_1^2 f = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}.$$

As we will see below, it will be possible to separate out the angular dependences and to reduce the oscillation equations to ordinary differential equations with one independent variable r. For variable separation, we will use normalized spherical harmonics

$$Y_{\ell m}(\theta,\varphi) = (-1)^m \left[\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!} \right]^{\frac{1}{2}} P_\ell^m(\cos\theta)\, e^{im\varphi}, \tag{6.4}$$

where $P_\ell^m(\cos\theta)$ are the associated Legendge polynomials

$$P_\ell^m(z) = \frac{(1-z^2)^{m/2}}{\ell!\, 2^\ell} \frac{d^{(\ell+m)}}{dz^{(\ell+m)}} (z^2-1)^\ell, \qquad z = \cos\theta. \tag{6.5}$$

The degree ℓ of the spherical harmonic takes any integer value $\ell = 0, 1, 2, \ldots$; at each ℓ, the azimuthal order m can take $2\ell + 1$ value, $m = -\ell, \ldots, \ell$. Spherical harmonics satisfy the second-order partial differential equation

$$\nabla_1^2 Y_{\ell m} + \ell(\ell + 1) Y_{\ell m} = 0. \tag{6.6}$$

6.2 Linear adiabatic wave equations in Cowling approximation

As with radial oscillations, we assume that the oscillations are adiabatic, and start with the same equations for linear perturbations (2.19, 2.21). To simplify the analysis, we will neglect in this lecture the effects of gravity perturbations; in stellar pulsation theory this approximation is known as Cowling approximation. This approximation is not very restrictive and can in general be easily relaxed; we adopt it here with the only reason to make mathematical derivations more transparent. In Cowling approximation, our starting equations are

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -\nabla p' - \rho' \nabla \psi_0, \tag{6.7a}$$

$$\frac{\partial \rho'}{\partial t} = -\nabla \cdot (\rho_0 \mathbf{u}), \tag{6.7b}$$

$$\frac{\partial p'}{\partial t} + \mathbf{u} \cdot \nabla p_0 = \frac{\Gamma_1 p_0}{\rho_0} \left(\frac{\partial \rho'}{\partial t} + \mathbf{u} \cdot \nabla \rho_0 \right). \tag{6.7c}$$

We now seek the solutions of these equations with sinusoidal time dependence and with angular dependence specified by a particular spherical harmonic as

$$\delta\mathbf{r} = \left[\hat{\mathbf{r}}\,U(r)Y_{\ell m}(\theta,\varphi) + V(r)\nabla_1 Y_{\ell m}(\theta,\varphi)\right]e^{i\omega t},$$

$$p' = p_1(r)\,Y_{\ell m}(\theta,\varphi)\,e^{i\omega t},$$

$$\rho' = \rho_1(r)\,Y_{\ell m}(\theta,\varphi)\,e^{i\omega t}, \qquad (6.8)$$

with $\mathbf{u} = \partial\delta\mathbf{r}/\partial t = i\omega\delta\mathbf{r}$. The expressions (6.8) allow to separate the angular dependences in equations (6.7), reducing them to the system of ordinary differential equations

$$-\rho_0\omega^2 U = -\frac{d}{dr}p_1 - \rho_1 g_0, \qquad (6.9a)$$

$$-\rho_0\omega^2 V = -\frac{1}{r}p_1, \qquad (6.9b)$$

$$\rho_1 = -\frac{1}{r^2}\frac{d}{dr}\left(r^2\rho_0 U\right) + \frac{\ell(\ell+1)}{r}\rho_0 V, \qquad (6.9c)$$

$$p_1 - \rho_0 g_0 U = c^2\left(\rho_1 + \frac{d\rho_0}{dr}U\right), \qquad (6.9d)$$

using $d\psi_0/dr = g_0$, $dp_0/dr = -\rho_0 g_0$ and $\Gamma_1 p_0/\rho_0 = c^2$. The first of the equations (6.9) comes from the radial component of the momentum equation (6.7a), the second - from its horizontal component. Eliminating ρ_1 and V in equations (6.9), we arrive to

$$\frac{dU}{dr} = \left(\frac{g_0}{c^2} - \frac{2}{r}\right)U - \frac{1}{\rho_0 c^2}\left(1 - \frac{\ell(\ell+1)c^2}{r^2\omega^2}\right)p_1, \qquad (6.10a)$$

$$\frac{dp_1}{dr} = \left(\omega^2 - N^2\right)\rho_0 U - \frac{g_0}{c^2}p_1, \qquad (6.10b)$$

where N stands for the Brunt-V�is�l� frequency defined earlier by equation (5.5).

Exercise 6.1 Fill in the missing steps to derive the oscillation equations (6.10), starting with variable separation in (6.7) by using equations(6.1-6.6).

6.3 Boundary conditions

The boundary conditions are derived in a manner similar to radial equations. At $r = 0$, the second-order system of linear differential equations (6.10) has a regular singular point; one of the two linearly-independent solutions is regular, another is singular. Near the origin, the physically relevant regular solution behaves as

$$U(r) \simeq \ell \, a_0 r^{\ell-1}, \quad p_1(r) \simeq \rho_0(0)\omega^2 a_0 r^{\ell}. \quad (6.11)$$

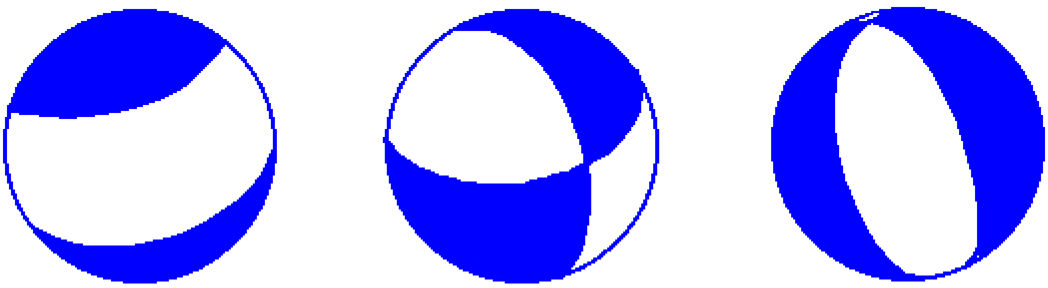
At the stellar surface, with Lagrangian pressure perturbation set to zero, the outer boundary condition is

$$p_1 - \rho_0 g_0 U = 0, \qquad r = R, \qquad (6.12)$$

the same as for radial modes.

6.4 Mode classification in degree ℓ.

As with radial oscillations, the boundary-value problem specified by the differential equations (6.10) and boundary conditions (6.11, 6.12) has non-trivial solutions only for certain values of ω^2, called eigenvalues. The major difference is that we now have a separate set of eigenfrequencies for each particular value of degree ℓ. The spherical harmonic $Y_{\ell m}(\theta, \varphi)$ determines the angular dependence of the eigenfunctions, and hence the surface distribution of the oscillation amplitudes as seen by an observer. Radial oscillations is just a particular case with $\ell = 0$. Oscillations with $\ell = 1$ are called dipole oscillations, $\ell = 2$ - quadrupole, etc. The surface amplitudes of $\ell = 2$ modes are shown schematically below. Dark and light areas correspond to $Y_{\ell m}(\theta, \varphi)$ positive and negative (*e.g.* when dark areas move upwards, light areas move downwards):



$\ell = 2, m = 0 \qquad \ell = 2, m = 1 \qquad \ell = 2, m = 2$

The degree ℓ is the total number of the node lines $Y_{\ell m}(\theta, \varphi) = 0$ on the solar surface. The azimuthal order m is the number of the node lines going along a meridian; there are $\ell - m$ node lines parallel to the equator. Note that m does not enter the oscillation equations (6.10), which means that modes of the same ℓ but with different m have the same frequencies. This degeneracy comes from the spherical symmetry of the equilibrium solar configuration.

6.5 Local dispersion relation. Mode classification in radial order n.

Assume that we have solutions of the wave equations (6.10) which oscillate rapidly in radial direction,

$$U(r) = \bar{U}(r)\exp(ik_r r), \quad p_1(r) = \bar{p}_1(r)\exp(ik_r r) \quad (6.13)$$

with amplitudes $\bar{U}(r)$ and $\bar{p}_1(r)$ varying much slower than the exponents (k_r is large). The analysis similar to that of radial modes now leads to the dispersion relation

$$k_r^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\ell(\ell+1)}{r^2}\frac{c^2}{\omega^2}\right)\left(1 - \frac{N^2}{\omega^2}\right). \qquad (6.14)$$

We observe that when $\ell \neq 0$, there are now two possibilities of having k_r^2 large and positive: ω^2 shall be either very large, when we have approximately

$$k_r^2 = \frac{\omega^2}{c^2} - \frac{\ell(\ell+1)}{r^2}, \qquad (6.15)$$

or very small, when we have approximately

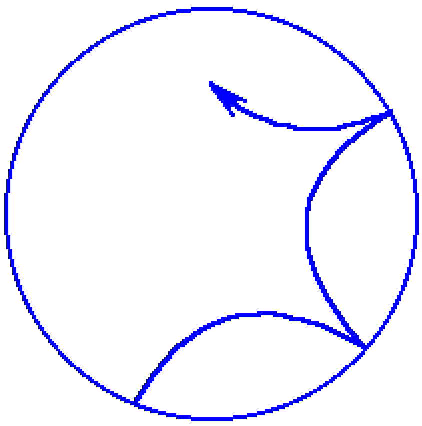
$$k_r^2 = \frac{\ell(\ell+1)}{r^2}\left(\frac{N^2}{\omega^2} - 1\right). \qquad (6.16)$$

The physics behind this result is that we now have two different types of waves which can propagate in the solar interior. The last relation (6.16) refers to internal gravity waves, with restoring forces provided by buoyancy. These waves are known in the oceans, and in the Earth's atmosphere. The gravity waves can be "trapped" to form the low-frequency standing waves, called gravity modes, or g-modes.

At high frequencies, we have (6.15) as a proper dispersion relation, which can be rewritten as

$$k_r^2 + k_h^2 = \frac{\omega^2}{c^2}, \quad k_h = \frac{\sqrt{\ell(\ell+1)}}{r}. \qquad (6.17)$$

Here k_h can be interpreted as a horizontal wavenumber, *i.e.* the horizontal component of the total wavevector $\mathbf{k}$. We have nothing else but the dispersion relation of the sound waves $k^2 = \omega^2 / c^2$. When $\ell = 0$, we have radial waves. A non-radial wave propagation and trapping of the acoustic wave in solar interior are shown schematically below. When travelling downwards,the acoustic wave suffers the refraction because of the larger temperature, and hence larger sound speed deeper in the Sun, reflects back after approaching the surface, and so on. If after a closed path the wave returns back in a proper phase, a standing wave is formed. This standing wave represents a particular mode of solar oscillations. These high-frequency modes are called acoustic modes, or p-modes. At given degree ℓ, different p modes are classified according to the number of nodes in their radial displacement function $U(r)$; the number of nodes is called the radial order n. Mode p_1 has one node in $U(r)$, mode p_2 has two nodes, etc.



The frequencies of p modes increase when the radial order n increases; frequencies of g modes decrease when their radial order increases. Between the two frequency domains of the high-frequency p modes and the low-frequency g modes there is usually an extra mode, which physical nature is that of a surface gravity wave: this is the so-called fundamental, or f-mode. A more comprehensive discussion of the classification of nonradial oscillations can be found in [6.1].

6.6 Inversion of the sound speed in solar interior

In solar seismology we have precise measurements of a large number of p-mode frequencies of different degree ℓ (from zero to few thousands) and of different radial oreder n. These observational data allow to address the inverse problem of solar seismology - the reconstruction of the internal structure of the Sun from its oscillation frequencies.

Using the local dispersion analysis of the previous section, the resonant frequencies must satisfy

$$\int_{r_1}^{R} k_r(r)\,dr = \pi(n+\alpha), \qquad (6.18)$$

where r_1 is an inner turning point with $k_r(r_1) = 0$. Equation (6.18) states that at frequencies of acoustic resonances, integer number of half-waves in $U(r)$ shall fit within the acoustic cavity $r_1 < r < R$; this number is radial order n. An additional phase shift α, which we will not specify explicitly here, accounts for a proper boundary conditions at the top and at the bottom of the acoustic cavity.

With radial wavenumber k_r specified by equation (6.15), we have

$$F(\tilde{w}) = \pi\frac{n+\alpha}{\omega}, \qquad (6.19)$$

$$F(\tilde{w}) = \int_{r_1}^{R}\left(\frac{r^2}{c^2} - \tilde{w}^2\right)^{1/2}\frac{dr}{r}, \qquad (6.20)$$

where

$$\tilde{w}^2 = \frac{\ell(\ell+1)}{\omega^2}. \qquad (6.21)$$

The oscillation frequencies provide the right-hand side of the equation (6.19), and thus function $F(\tilde{w})$ is available from the observations. From its definition (6.20), $F(\tilde{w})$ is determined by the sound-speed profile $c(r)$ only. The integral relation (6.20) between the two functions $F(\tilde{w})$ and $c(r)$ can be inverted analytically. The result is

$$\ln\frac{r_1}{R} = \frac{2}{\pi}\int_{r_1/c_1}^{R/c_s}\left(\tilde{w}^2 - \frac{r_1^2}{c_1^2}\right)^{-1/2}\frac{dF}{d\tilde{w}}\,d\tilde{w}, \qquad (6.22)$$

where $c_1 = c(r_1)$ and $c_s = c(R)$. With expression (6.22), we obtain r_1 as a function of r_1 / c_1, and hence c as a function of r.

Exercise 6.2 Prove the equation (6.22). [Hint: differentiate $F(\tilde{w})$ as given by (6.20), substitute the resulted integral into (6.22), and change the order of integration]. This exercise is an optional one: try it only if you feel yourself confident with double integrals.

LITERATURE

[6.1] Unno, W., Osaki, Y., Ando, H., Saio, H. & Shibahashi, H., 1989. *Nonradial oscillations of stars* (2nd edition) (University of Tokyo Press).