FI 2201 Electromagnetism

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Techniques in solving Electric Potentials

LAPLACE'S EQUATIONS IN CARTESIAN COORDINATE SYSTEM

Introduction

- Our primary task of electrostatics is to find the electric field of a given stationary charge distribution.
- In principle we can calculate this using Coulomb's law

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_v \frac{\rho(\vec{r}')}{r^2}\hat{r}\,d\tau'$$

- Unfortunately, the above integral can be difficult to calculate for arbitrary charge distribution also the problem increases since we need to worry about the direction of the field.
- Occasionally, for problem with high symmetry, we can use Gauss' Law

$$\oint_S \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$$

Introduction

- A better strategy is to calculate the electric potential which is somewhat easier integral

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_v \frac{\rho(\vec{r}')}{r}\,d\tau'$$

  and the electric field can be calculated from the gradient of this electric potential.
- However, in some problem, we only know the total charge, while the charge distribution is not known (for example in a conductor).
- Or in some cases, we only know the potential at certain boundary.
- For this kind of problem, recasting the above integral equation into a differential equation is a better choice.

Introduction

- The differential equation of the potential problem is the Poisson's equation

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

  with appropriate boundary conditions, the solution of this problem can be obtained.
- When, the region of interest contains no charge distribution (the charge distribution that resulted the electric potential we are looking for is located outside the region of interest), then the above equation reduces to Laplace's equation

$$\nabla^2 V = 0$$

- In Cartesian coord. system : $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$

Laplace's Equation in 1D

- In 1D problem, the Laplace's equation simplifies into

$$\frac{d^2 V}{dx^2} = 0$$

- With a general solution given as

$$V(x) = mx + b$$

  i.e. an equation of a straight line and it contains two undetermined constants, just as what we expected from a second-order differential equation.
- These undetermined constants are fixed by the given boundary condition.

$$\left.\begin{aligned} V(x=1) &= 4 \\ V(x=5) &= 0 \end{aligned}\right\} \rightarrow \quad V(x) = -x + 5$$

Laplace's Equation in 1D

- Two features of this solution are
  - For any a, $V(x)$ is the average of $V(x + a)$ and $V(x - a)$

$$V(x) = \frac{1}{2}\left[V(x+a) + V(x-a)\right]$$

  - Laplace's equation tolerates *no local maxima or minima*, extreme value of V must occur at the boundary (end) points. This is a consequences of the first property above.

Laplace's Equation in 2D

- In 2D problem, the Laplace's equation becomes into

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

- For this partial differential equation, we cannot write a closed form general solution.
- However, the two features of the solution are the same
  - For any point (x,y), $V(x,y)$ is the average of those potential on a circle with radius R around it

$$V(x,y) = \frac{1}{2\pi R}\oint_{circle} V d\ell$$

  - Laplace's equation tolerates *no local maxima or minima*, extreme value of V must occur at the boundary points.
- Functions with these properties (i.e. solution to the Laplace's equation) are called a harmonic function.

Laplace's Equation in 3D

- Two features of the solution remains the same
  - The value of $V(\vec{r})$ at point $\vec{r}$, is the average value of $V(\vec{r})$ over a spherical surface of radius R centered at $\vec{r}$

$$V(\vec{r}) = \frac{1}{4\pi R^2} \oint_{sphere} V da$$

  - Laplace's equation tolerates *no local maxima or minima*, extreme value of V must occur at the boundary (end) points. This is a consequences of the first property above.
- The proof of the second property above is as follows, consider a point charge q and we want to find the potential a distance away from this charge. For simplicity, we put the point charge at the z axis and calculate the potential at the origin through averaging around a sphere.

Laplace's Equation in 3D

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

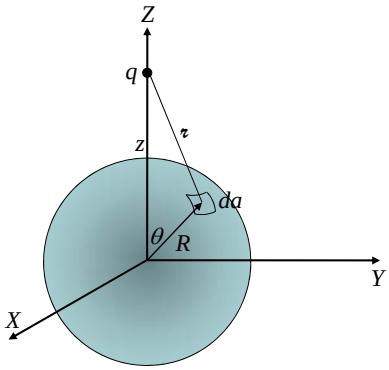
$$r^2 = z^2 + R^2 - 2zR\cos\theta$$

$$V(0) = \frac{1}{4\pi R^2} \oint_{sphere} V da$$

$$= \frac{1}{4\pi R^2}\frac{q}{4\pi\varepsilon_0}\int \frac{R^2 \sin\theta \, d\varphi \, d\theta}{\sqrt{z^2 + R^2 - 2zR\cos\theta}}$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{2zR}\sqrt{z^2 + R^2 - 2zR\cos\theta}\Big|_0^{\pi}$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{2zR}\left[(z+R) - (z-R)\right] = \frac{1}{4\pi\varepsilon_0}\frac{q}{z}$$



Boundary Conditions and Uniqueness

- To solve for the potential, we need boundary conditions.
- For the 1D problem, it is easy:
  - Specify the potential values at both ends, or
  - Specify the potential value at one end and the value of the derivatives of the potential at the other end
- But we cannot use the following boundary conditions:
  - Specify just one boundary the value or the value of the derivative of the potential.
  - Specify the value of the derivative of the potential at both ends (redundant if they are equal, inconsistent if they are different)
- For the 2D and 3D problem, which boundary condition should we use ?

Boundary Conditions and Uniqueness

The First Uniqueness Theorem

- The solution to Laplace's equation in some volume $\mathcal{V}$ is uniquely determined if the potential is specified on the boundary surface $\mathcal{S}$.
- Proof: suppose we have V_1 and V_2 two solutions of the same boundary condition, then both of them satisfy Laplace's equation

$$\nabla^2 V_1 = 0 \quad and \quad \nabla^2 V_2 = 0$$

- The difference $V_3 = V_1 - V_2$, also satisfies Laplace's eqn.
- But V_3 is zero at the boundary, and Laplace's equation admits no local extrema in the volume of interest (extrema only at the boundary), hence V_3 is zero everywhere, thus

$$V_3 = 0 \quad \rightarrow \quad V_1 = V_2$$

Boundary Conditions and Uniqueness

- In other words, the First Uniqueness theorem also states that, no matter how you got your results for the potential, as long as they satisfy Laplace's equation and has the correct value at the boundary, then you result is correct.
- The uniqueness theorem can also include the problem when there are some charge distribution on the volume of interest.
- Suppose we have V_1 and V_2 two solutions of the Poison's equation with the same boundary condition,

$$\nabla^2 V_1 = -\frac{\rho}{\varepsilon_0} \quad and \quad \nabla^2 V_2 = -\frac{\rho}{\varepsilon_0}$$

Boundary Conditions and Uniqueness

- Then the difference will satisfies Laplace's equation

$$\nabla^2 (V_1 - V_2) = \nabla^2 V_3 = 0$$

and hence by the same argument as before, it follows that on the boundary V_3 is zero, hence

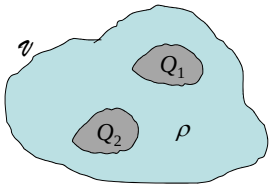
$$V_3 = 0 \quad \rightarrow \quad V_1 = V_2$$

Boundary Conditions and Uniqueness

- For problems with charged conductors, although we do not know how these charges is distributed in the conductor, but the following uniqueness theorem guarantee that the electric field is uniquely determined.

Second Uniqueness Theorem

- In a volume $\mathcal{V}$, which may contain some charge density ρ, surrounding some charged conductors, the electric field is uniquely determined if the total charge on each conductors is given.
- Proof: see textbook.



Laplace's Equation in Cartesian Coord.

- To solve the 3D Laplace's equation, we apply the method of separation of variables: we look for solutions that are product of functions, each of which depends on only one of the coordinates.
- For this method to be applicable, the boundary conditions should also be written as product of boundary conditions, i.e. the boundary condition does not mixed the coordinates of the problem.
- This method is applicable to problems where potential or the surface charge density σ on the boundary is given.

- **Example 3.3**

Complete and Orthonormal functions

- The set of trigonometric functions sine and cosine forms a complete set of function.
- The set of functions $f_n(x)$ is said to be complete if any other function $f(x)$ can be expressed as a linear combination of these complete set of functions

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$$

In other words, the complete set of functions are basis function in the functional space.

- A more useful complete set of functions are the complete set of orthogonal functions, with its orthogonality is defined by an inner product operation

$$\int_0^a \beta(x) f_n(x) f_{n'}(x) dx = \delta_{n,n'}$$

Laplace's Equation in Cartesian Coord.

- **Example 3.5**
- (see also **Example 3.4**)