SpaceTime and Gravity: Assignment 2 Solutions

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February 12, 2009

In what follows, unless otherwise stated, we will use units such that the speed of light, $c = 1$.

1.

For angle θ measured in radians we have $r = a\theta$ and $\sin(\theta) = \frac{x}{a}$. We want to obtain the circumference of the inner circle of radius x, i.e. $2\pi x$.

So

$$\begin{aligned}
\frac{x}{a} &= \sin(\theta) \quad (1)\\
x &= a\sin(\theta) \quad (2)\\
&= a\sin(\frac{r}{a}) \quad (3)\\
2\pi x &= 2\pi a\sin(\frac{r}{a}) \quad (4)\\
c &= 2\pi a\sin(\frac{r}{a}) \quad (5)
\end{aligned}$$

as required.

So why is the value of π wrong in the Bible? We imagine that the cast bronze was built on a curved surface as in the diagram. To obtain the correct value of π we would need to take the x variable as our radius, not the actual "rim to rim" distance which they took. However we can estimate the radius of the curved surface the bronze was built on to obtain the seemly incorrect value of π. We are given the circumference of the circle $c = 30$ cubits and the "rim to rim" radius $2r = 10$ cubits. So, by expanding $\sin(\theta)$ in a power series,

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ..... \quad (6)$$

we obtain

$$
\begin{aligned}
c &= 2\pi a \sin(\frac{r}{a}) && (7)\\
&= 2\pi a \left(\frac{r}{a} - \frac{r^3}{3!a^3} + ... \right) && (8)\\
\Rightarrow a &= \left(\frac{6(2\pi r - c)}{2\pi r^3} \right)^{-\frac{1}{2}} && (9)\\
&= \sqrt{\frac{2\pi r^3}{6(2\pi r - c)}} && (10)\\
&= \sqrt{\frac{250\pi}{6(10\pi - 30)}} && (11)\\
&= 9.61 cubits && (12)
\end{aligned}
$$

It is the notion of intrinsic curvature which is key here. Curvature is not something some outside obersver has to tell you about, you can do local experiments to determine the curvature of the space you are in. It is intrinsic to the space in question.

2.

We are given the definition of the variable

$$z = x + \tau y \tag{13}$$

where

$$\tau = \tau_1 + i\tau_2 \tag{14}$$

and we are asked to compute the line element

$$ds^2 = dz d\overline{z} \tag{15}$$

So

$$
\begin{aligned}
ds^2 &= dz d\overline{z} && (16)\\
&= (dx + \tau dy)(dx + \overline{\tau} dy) && (17)\\
&= dx^2 + (\tau_1 + i\tau_2) dy dx + (\tau_1 - i\tau_2) dy dx + (\tau_1^2 + \tau_2^2) dy^2 && (18)\\
&= dx^2 + 2\tau_1 dy dx + |\tau|^2 dy^2 && (19)
\end{aligned}
$$

[For the interested reader. So why is it important to sometimes introduce complex coordinates? Here is an example: Imagine you are given a line element of the form

$$ds^2 = dt^2 + dx^2 \tag{20}$$

then if we substitute the complex coordinates $z = Ae^{(t+ix)}$ and $\overline{z} = Ae^{(t-ix)}$ then the line element becomes

$$ds^2 = A^2 e^{2t} dz d\overline{z} \tag{21}$$

And by a convenient choice of A we can put this into the form $ds^2 = dz d\overline{z}$. This is amazing! We see that we can map the infinite past $t = -\infty$ to a point on a two dimensional plane $z = \overline{z} = 0$ and the rest of the temporal coordinate to a radial one. This is the first step in the conformal field theory of the propagating closed string!]

3.
We are given the flat space Minkowski line element

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{22}$$

and we are asked to perform a coordinate substitution of the form

$$\begin{aligned} x &= r\sin(\theta)\cos(\phi) && (23) \\ y &= r\sin(\theta)\sin(\phi) && (24) \\ z &= r\cos(\theta). && (25) \end{aligned}$$

Note that these are 3d spherical polar coordinates so we expect our final line element to have some degree of spherical symmetry. So

$$\begin{aligned} dx &= dr\sin(\theta)\cos(\phi) + r\cos(\theta)\cos(\phi)d\theta - r\sin(\theta)\sin(\phi)d\phi && (26) \\ dy &= dr\sin(\theta)\sin(\phi) + r\cos(\theta)\sin(\phi)d\theta + r\sin(\theta)\cos(\phi)d\phi && (27) \\ dz &= dr\cos(\theta) - r\sin(\theta)d\theta && (28) \end{aligned}$$

and

$$\begin{aligned} dx^2 &= dr^2\sin^2(\theta)\cos^2(\phi) + 2drr\cos(\theta)\sin(\theta)\cos^2(\phi)d\theta - 2rdr\sin^2(\theta)\sin(\phi)\cos(\phi)d\phi \\ &\quad + r^2\cos^2(\theta)\cos^2(\phi)d\theta^2 - 2r^2\sin(\theta)\cos(\theta)\cos(\phi)\sin(\phi)d\theta d\phi \\ &\quad + r^2\sin^2(\theta)\cos^2(\phi)d\phi^2 \\ dy^2 &= dr^2\sin^2(\theta)\sin^2(\phi) + 2rdr\sin(\theta)\cos(\theta)\sin^2(\phi)d\theta + 2rdr\sin^2(\theta)\sin(\phi)\cos(\phi)d\phi \\ &\quad + r^2\cos^2(\theta)\sin^2(\phi)d\theta^2 + 2r^2\cos(\theta)\sin(\theta)\cos(\phi)\sin(\phi)d\theta d\phi \\ &\quad + r^2\sin^2(\theta)\cos^2(\phi)d\phi^2 \\ dz^2 &= dr^2\cos^2(\theta) - 2rdr\cos(\theta)\sin(\theta)d\theta + r^2\sin^2(\theta)d\theta^2 \end{aligned}$$

Which means

$$\begin{aligned} dx^2 + dy^2 + dz^2 &= dr^2\sin^2(\theta) + 2drr\cos(\theta)\sin(\theta)d\theta + r^2\cos^2(\theta)d\theta^2 + r^2\sin^2(\theta)d\phi^2 \\ &\quad + dr^2\cos^2(\theta) + r^2\sin^2(\theta)d\theta^2 - 2drr\cos(\theta)\sin(\theta)d\theta \\ &= dr^2 + r^2d\theta^2 + r^2\sin^2(\theta)d\phi^2 \end{aligned}$$

Which is the spherically symmetric line element.
Finally we perform the light-cone transformations so that

$$\begin{aligned} u &= t - r && (29) \\ v &= t + r && (30) \end{aligned}$$

Inverting these relations yields

$$\begin{aligned} dt &= \frac{du + dv}{2} && (31) \\ dr &= \frac{dv - du}{2} && (32) \end{aligned}$$

So

$$
\begin{aligned}
-dt^2 &= -\frac{1}{4}(du^2 + dv^2 + 2dudv) && (33) \\
dr^2 &= \frac{1}{4}(dv^2 + du^2 - 2dudv) && (34)
\end{aligned}
$$

Which yields

$$
\begin{aligned}
ds^2 &= -\frac{1}{4}(du^2 + dv^2 + 2dudv) + \frac{1}{4}(dv^2 + du^2 - 2dudv) + \frac{1}{4}(v^2 + u^2 - 2vu)d\theta^2 \\
&\quad + \frac{1}{4}(v^2 + u^2 - 2vu)\sin^2(\theta)d\phi^2 \\
&= -dudv + \frac{(u^2 + v^2 - 2uv)}{4}(d\theta^2 + \sin^2(\theta)d\phi^2).
\end{aligned}
$$

u and v are called light-cone coordinates because they "sit" on the light-cone. Plotting the t versus r graph we see that u and v parametrize the light cone of the theory.

1 Summary of Important points

1.Curvature is an intrinsic property of any space.
2.Co-ordinate transformations only change the form of the line-element, not the physics it describes.