Name .... Course ....
Print and use this sheet in conjunction with ***MathinSite***'s 'Maclaurin Series' applet and worksheet.

Maclaurin Series

Learning Outcomes

After reading this theory sheet, you should

- recognise the difference between a function and its polynomial expansion (if it exists!)
- understand what is meant by a series approximation
- be able to find series expansions (Maclaurin Series) for algebraic functions
- be able to find series expansions for certain transcendental functions
- be able to use Maclaurin Series to find approximate values of functions

From Functions to Polynomial Series Expansions

A **Polynomial Series** is a mathematical expression consisting of added terms, terms which consist of a constant multiplier and one or more variables raised to integral powers. For example, $3x^2 - 2x + 7$ and $5y + 8x^3z$ are **closed** polynomials (i.e. polynomials containing a finite number of terms).

Functions of the type $(1 + x)^n$ and their Polynomial Representation

Under certain conditions mathematical **functions** can be equated to polynomial series - a simple example being the quadratic function $f(x) = (x + 1)^2$, in which the brackets can be expanded giving $f(x) = x^2 + 2x + 1$. Here $x^2 + 2x + 1$ is a polynomial series.

When considering polynomial expansions of this type, the order is usually reversed, i.e.

$$(1+x)^2 = 1+2x+x^2$$

Similarly

$$(1+x)^3 = 1+3x+3x^2+x^3$$

$$(1+x)^4 = 1+4x+6x^2+4x^3+x^4$$

$$(1+x)^5 = 1+5x+10x^2+10x^3+5x^4+x^5 \text{, etc}$$

These expansions are special cases of the more general ***Binomial Theorem***. Namely,

$$\boxed{(1+x)^n = 1+nx+n(n-1)\frac{x^2}{2!}+n(n-1)(n-2)\frac{x^3}{3!}+n(n-1)(n-2)(n-3)\frac{x^4}{4!}+\dots}$$

where the '+…' indicates an **open**-ended polynomial (i.e. one which continues forever) and 2! (2 factorial) means 2 x 1; 3! (3 factorial) = 3 x 2 x 1; etc.

For a *positive integer* value of n, application of the Binomial Theorem results in **closed** polynomials. For example, with $n = 2$, the above Theorem gives, as expected,

$$(1+x)^2 = 1+2x+x^2 \qquad \text{(Try it!)}$$

For a negative integer value of n, or a fraction value of n this will result in an open polynomial. For example, with $n = -1$, the above Binomial Theorem results in

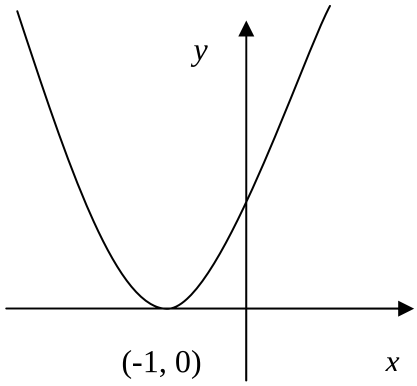
$$(1+x)^{-1} = 1+(-1)x+(-1)(-2)\frac{x^2}{2!}+(-1)(-2)(-3)\frac{x^3}{3!}+(-1)(-2)(-3)(4)\frac{x^4}{4!}+\dots$$

$$= 1-x+x^2-x^3+x^4+\dots$$

(Try it! Also, note the pattern of successive terms)

Let's consider these examples in more detail. The two **functions**, $f(x) = (1 + x)^2$ and $f(x) = (1 + x)^{-1}$ are each being equated to a **polynomial series**.

Plotting $f(x) = (1 + x)^2$, a quadratic function, results in a parabolic graph; fundamentally, $y = x^2$ shifted by 1 unit to the left, i.e. with its vertex at (-1, 0).



Now, since $f(x) = (1+x)^2 = 1+2x+x^2$, ***exactly the same parabola will result*** for $f(x) = 1+2x+x^2$

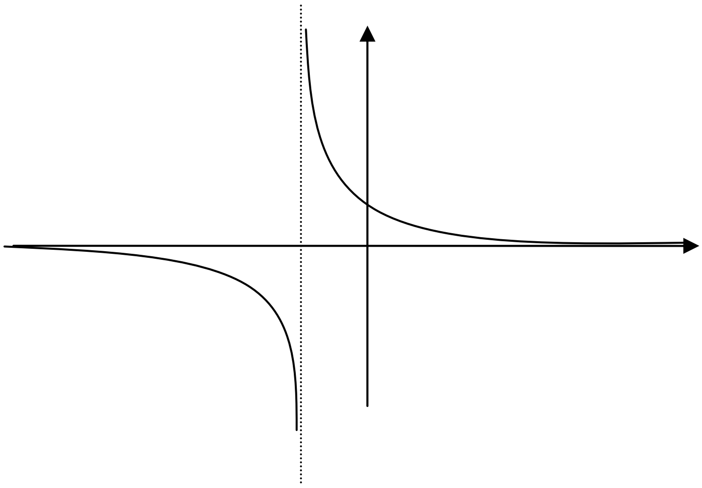
Note that the function could be approximated by incorporating successive terms of the polynomial expansion, so that

$$(1+x)^2 \approx 1$$

$$(1+x)^2 \approx 1+2x$$ ("$\approx$" – "is approximately equal to")

Plotting either of these approximations is obviously not going to give anything like a parabola since the first is a horizontal line through $y = 1$ and the second is a straight line gradient 2 and y-intercept 1. Only the full three-term expansion will produce the correct parabolic curve.

Plotting the reciprocal function $f(x) = (1 + x)^{-1} = 1 / (1 + x)$, results in the graph of a rectangular hyperbola; fundamentally, $y = 1/x$ shifted by 1 unit to the left, i.e. with $x = -1$ a vertical asymptote and $y = 0$ a horizontal asymptote.



Bringing in successive terms for this function results in

$$(1+x)^{-1} \approx 1$$

$$(1+x)^{-1} \approx 1-x$$

$$(1+x)^{-1} \approx 1-x+x^2$$

etc

The first two of these are straight line approximations, so do not even curve in the way that the function does. At least the quadratic approximation produces a curve. Note that all three approximations above (and in fact all higher order approximations) have a y-intercept of $y = 1$ – as does the function itself.

Use the accompanying applet on this example to see the effects described above.

Bearing in mind that the polynomial expansion of this function is an infinite series, it would take an infinite number of terms to produce a true polynomial approximation of this function. Erm, well yes, you would think so. Unfortunately, the polynomial expansion of $f(x) = (1 + x)^{-1}$ only matches the true function curve in the range $-1 \le x < +1$ *no matter how many terms are included*. Outside that range of values the polynomial approximation either does not exist or is extremely inappropriate.

Again, these effects can be seen using the accompanying applet.

Already then, it can be seen that

- some functions can be approximated successfully for *all* values of x
- some give a good approximation only over a restricted range and
- some do not have a polynomial representation at all (as will be seen later).

Using the Binomial Expansion gives polynomial representations of functions of the type $(1 + x)^n$. But is it possible to represent other types of function (sine, log, exponential, for example - so-called ***transcendental*** functions) as polynomial series. The answer is yes – but not always!

Transcendental Functions and their Polynomial Representation

It takes a great leap of faith to accept that it is going to be possible to represent a transcendental function such as sin x in terms of an algebraic polynomial expansion – but this is exactly what Maclaurin Series analysis allows us to do.

So, is it possible to expand *any* function $f(x)$ as a power series? Yes, possibly, but note all the assumptions (below) that have to be made on the way!

Assume that a function, $f(x)$, can be expanded as

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \ldots$$

Here the ascending powers of x show the function expanded as a polynomial series. This series has, at present, undetermined coefficients (constants that multiply successive terms in the series), $a_0, a_1, a_2, a_3, a_4, a_5, a_6 \ldots$

Now, putting $x = 0$ in the above series, ***assuming*** the function *exists* at $x = 0$ (watch out, functions like $f(x) = 1/x$ don't!), gives

$$f(0) = a_0 \qquad \boxed{\text{or } a_0 = f(0)}$$

Next, ***assuming*** that the function and its polynomial representation are differentiable, then

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \ldots$$

Putting $x = 0$ in *this* series, ***assuming*** the *differential of the function exists* at $x = 0$, gives

$$f'(0) = a_1 \qquad \boxed{\text{or } a_1 = f'(0)}$$

Next, ***assuming*** that the function and its polynomial representation are now twice differentiable, then

$$f''(x) = 2a_2 + 3\times 2a_3 x + 4\times 3a_4 x^2 + 5\times 4a_5 x^3 + 6\times 5a_6 x^4 + \ldots$$

Putting $x = 0$ in *this* series, ***assuming*** the *second differential of the function exists* at $x = 0$, gives

$$f''(0) = 2\times 1\times a_2 \qquad \boxed{\text{or } a_2 = \frac{f''(0)}{2!}}$$

$$\text{Similarly, } \boxed{a_3 = \frac{f'''(0)}{3!}\ ,\ a_4 = \frac{f^{(iv)}(0)}{4!},\ a_5 = \frac{f^{(v)}(0)}{5!}, \ldots}$$

Note the pattern involved here.

So, with the unknown coefficients found in terms of the value of the function and its derivatives at $x = 0$, the general Maclaurin Theorem can be written as:

The polynomial series representation (i.e. ***Maclaurin Series***) of any infinitely differentiable function, *f*(*x*), whose value, and the values of all of its derivatives, exist at *x* = 0 is given by the infinite series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(iv)}(0)}{4!}x^4 + \frac{f^{(v)}(0)}{5!}x^5 + \frac{f^{(vi)}(0)}{6!}x^6 + \ldots$$

… note the strict conditions about differentiability and existence at *x* = 0.

Maclaurin Series of sin x **(NOTE: *x* in *RADIANS*)**

Let	$f(x) = \sin x$	so $f(0) = \sin 0 = 0$
so	$f'(x) = \cos x$	so $f'(0) = \cos 0 = 1$
and	$f''(x) = -\sin x$	so $f''(0) = -\sin 0 = 0$
so	$f'''(x) = -\cos x$	so $f'''(0) = -\cos 0 = -1$
and	$f^{(iv)}(x) = \sin x$	so $f^{(iv)}(0) = \sin 0 = 0$

Note that the fourth derivative takes us back to the start point, so these values repeat in a cycle of four as 0, 1, 0, -1, 0, 1, 0, -1, 0, 1, 0, -1, 0, 1, 0, -1, etc

Note that the function is infinitely differentiable and that it, and all its derivatives, exist at *x* = 0.

Substitution of these values back into the Maclaurin Series gives

$$f(x) = \sin x = 0 + 1.x + 0.x^2 + -1.x^3 + 0.x^4 + 1.x^5 + 0x^6 + \ldots$$

or

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots}$$

When you think about it, this result is amazing! A transcendental, trigonometrical function being represented by an algebraic, polynomial function! Almost more amazing is how the series comes out in a regular form in which there is a pattern. There are no prizes for guessing that the next term would have been $x^9 / 9!$ and the next $- x^{11} / 11!$

Maclaurin Series of e^x^

Let	$f(x) = e^x$	so $f(0) = e^0 = 1$
so	$f'(x) = e^x$	so $f'(0) = e^0 = 1$
and	$f''(x) = e^x$	so $f''(0) = e^0 = 1$, etc

of course, the function and all its successive derivatives are the same, so these values repeat indefinitely as 1, 1, 1, 1, 1, 1, 1, 1, etc

Note again that the function was infinitely differentiable and that it, and all its derivatives, exist at *x* = 0.

Substitution of these values back into the Maclaurin Series gives

$$f(x) = e^x = 1 + 1.x + \frac{1}{2!}.x^2 + \frac{1}{3!}.x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \ldots$$

or $$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots}$$

Another truly amazing result! The exponential function can also be represented by a polynomial expansion with, again, a regular pattern. Again, there are no prizes for guessing subsequent terms!

Maclaurin Series of ln **x**

Let $f(x) = \ln x$ so $f(0) = \ln 0 = ?$

so $f'(x) = \dfrac{1}{x}$ so $f'(0) = \dfrac{1}{0} = ?$

and $f''(x) = -\dfrac{1}{x^2}$ so $f''(0) = -\dfrac{1}{0^2} = ?$, etc

Neither the function nor any of its derivatives exist at $x = 0$, ***so there is <u>no</u> polynomial Maclaurin expansion of the natural logarithm function ln x***.

Not wishing to be put off by this, is it possible to produce a Maclaurin series for *any* natural logarithm function at all? The answer is 'yes'. All that has to be done is to shift the function/curve left by 1 unit and an expansion for logarithm can be found since the new function and all its derivative now all exist at $x = 0$.

Maclaurin Series of ln **(1 + x)**

Let $f(x) = \ln(1+x)$ so $f(0) = \ln(1+0) = \ln 1 = 0$

so $f'(x) = \dfrac{1}{(1+x)}$ so $f'(0) = \dfrac{1}{1+0} = 1$

and $f''(x) = -\dfrac{1}{(1+x)^2}$ so $f''(0) = -\dfrac{1}{(1+0)^2} = -1$,

and $f'''(x) = \dfrac{2}{(1+x)^3}$ so $f'''(0) = \dfrac{2}{(1+0)^3} = 2$,

and $f^{(iv)}(x) = -\dfrac{3\times 2}{(1+x)^4}$ so $f^{(iv)}(0) = -\dfrac{3\times 2}{(1+0)^3} = -3\times 2$, etc

Substituting these values back into the general Maclaurin Series gives

$$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots}$$

note that there are no factorial values in the denominators.

IMPORTANT NOTE: Although the series for ln (1 + x) exists (owing to the function and its derivatives existing and the function and its derivatives being defined at $x = 0$, the function itself doesn't exist for $x \le -1$. In which case the function expansion does not exist for $x \le -1$ either. In fact, ***the series approximation is only valid within the 'narrow' range*** $-1 < x \le +1$ (as can be seen when using the accompanying Maclaurin Series applet).

***Maclaurin Series of cos* x**

The series for cos x, applying the same procedure is

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots}$$

This series, as with the sine series, is valid *for all* values of x (unlike the ln (1 +x) series), as is exhibited when using the accompanying Maclaurin Series applet.

Note, though, that there is a quick way of deriving the series for cos x if the series for sin x is already known. Since both sin x and cos x are both infinitely differentiable and their function and differential values all exist at $x = 0$, the Maclaurin Series for cos x could have been found by differentiating both sides of the series expansion for sin$x$ term by term. (Try it)

Note that, by applying the same procedure, differentiating the series for ln $(1 + x)$ term by term results in the series for $1 / (1 + x)$.

Maclaurin Series of inverse trigonometrical functions

Series for inverse trigonometrical functions can be complicated to find directly since successive differentials become unmanageable quite quickly. The idea of differentiability of functions *and their series* is useful in finding series of inverse trigonometrical functions.

Consider $f(x) = \sin^{-1} x$.

The differential of this function is $f'(x) = \dfrac{1}{\sqrt{1-x^2}} = \left(1-x^2\right)^{-1/2} = \left(1+\left(-x^2\right)\right)^{-1/2}$.

The polynomial series expansion for the last expression here can be found using the Binomial Theorem.

$$f'(x) = \left(1+\left(-x^2\right)\right)^{-1/2} = 1 + \left(-\tfrac{1}{2}\right)\left(-x^2\right) + \left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)\left(\frac{\left(-x^2\right)^2}{2!}\right) + \ldots = 1 + \tfrac{1}{2}x^2 + \tfrac{3}{8}x^4 + \ldots$$

Now reverse the differentiation process to obtain

$$\int f'(x)dx = \int \left(1+\left(-x^2\right)\right)^{-1/2} dx = \sin^{-1} x = \int 1 + \tfrac{1}{2}x^2 + \tfrac{3}{8}x^4 + \ldots\, dx$$

so $$\boxed{\sin^{-1} x = x + \tfrac{1}{6}x^3 + \tfrac{3}{40}x^5 + \ldots}$$

A comprehensive list of functions and their Maclaurin Series (including ranges of validity) can be found at http://mathworld.wolfram.com/MaclaurinSeries.html.

Using Maclaurin Series to Approximate Functions

A commonly used approximation for sin *x*, for <u>small</u> values of *x*, is $\sin x \approx x$. (Remember, *x* HAS to be in RADIANS!) Consider the following table.

Angle, x (^{o})	Angle, x (rads)	Sin x	$\lvert$% error$\rvert$
1	0.017453292	0.017452406	~ 0.005
2	0.034906585	0.034899496	~ 0.02
5	0.087266462	0.087155742	~ 0.13
10	0.174532925	0.173648177	~ 0.51
20	0.349065850	0.342020143	~ 2.06
90	1.570796327	1.000000000	~ 57.07
180	3.1415926536	0	∞(!)

Obviously as soon as the function value becomes zero (the denominator in the % error formula), the % error necessarily becomes infinite. This will happen every time *x* is a multiple of 180^{o}. Even so, notice that the approximation $\sin x \approx x$ is good up to about 10^{o} (< 1% error), but the further from zero the more inaccurate the approximate value becomes (e.g. there is a ~ 57% error in the 90^{o} approximation).

You should have noticed that the approximation $\sin x \approx x$ uses just the *first term* approximation of its Maclaurin Series. What happens to the accuracy if more terms are included? Consider the 90^{o} value again using the first 5 terms of the series.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \ldots$$

with x = 90^{o} ≡ $\frac{\pi}{2}$ = 1.570796327 radians, the series approximation gives

$$\sin\tfrac{\pi}{2} = \tfrac{\pi}{2} - \frac{\left(\frac{\pi}{2}\right)^3}{3!} + \frac{\left(\frac{\pi}{2}\right)^5}{5!} - \frac{\left(\frac{\pi}{2}\right)^7}{7!} + \frac{\left(\frac{\pi}{2}\right)^9}{9!} \ldots$$

so

$$\sin\tfrac{\pi}{2} = 1.570796327 - \frac{(1.570796327)^3}{3!} + \frac{(1.570796327)^5}{5!} - \frac{(1.570796327)^7}{7!} + \frac{(1.570796327)^9}{9!} \ldots$$

$$\sin\tfrac{\pi}{2} = 1.570796327 - 0.645964097 + 0.079692626 - 0.004681754 + 0.000160441 \ldots$$

giving

$$\sin\frac{\pi}{2} \approx 1.000003543 \qquad \text{(this time, a 0.0003543\% error)}$$

rather different to the 57% error obtained before!

Although this one example does not form a conclusive proof, it can be said that, in general, where a Maclaurin Series approximation is valid, the approximations are more accurate

- the closer the value of *x* is to zero, and
- the greater number of terms used from the Maclaurin Series.

So how many terms of a Maclaurin Series are needed? Well, as many as it takes to achieve the accuracy required.

You will have noticed in the last example that successive terms of the Maclaurin Series become smaller – the series converges. This will always be true for any x within the range of validity. So, to find the value of a function using its Maclaurin Series to a given accuracy, one only needs to use the number of terms that give the appropriate accuracy required. Consider the following example.

Example Use the Maclaurin Series for e^x to determine the value of $e^{0.5}$ to 3 d.p.

Now

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

so

$$e^{0.5} = 1 + 0.5 + \frac{0.5^2}{2!} + \frac{0.5^3}{3!} + \frac{0.5^4}{4!} + \frac{0.5^5}{5!} + \dots$$

i.e.

$$e^{0.5} = 1 + 0.5 + 0.125 + 0.0208\dot{3} + 0.0026041\dot{6} + 0.00026041\dot{6} + \dots$$

The last term here does not affect the third decimal place. In fact, the addition of *all* subsequent terms in the infinite series will *not* affect the third decimal place, so the value of $e^{0.5}$ to 3 d.p. can be found from adding just the first 5 terms ONLY. This gives $e^{0.5}$ = 1.648437499, or,

$$e^{0.5} = 1.648 \text{ (3 d.p.)}$$

Compare this with the value obtained directly from a calculator, $e^{0.5}$ = 1.6484721271 to 11 significant figures.

Try these!

1. All other things being equal, there are two main reasons why you would expect the Maclaurin Series for sin x to converge 'faster' than the Maclaurin series for ln x. What are these reasons?
2. Use as many terms of the relevant Maclaurin Series as is necessary to determine the values of sin 0.1 and ln 0.1 to 3 decimal places.
3. Use the accompanying Maclaurin Series applet to see how the series expansion for ln(1 + x) is inappropriate for x < -1 and x > +1.