Topic 25 — Diffraction and Resolution

L25.1 Resolution of a slit

We have seen that the central maximum of the diffraction pattern of a slit of width d illuminated with light of wavelength λ has the first minimum an angle λ/d away on each side.

Now suppose that light shines on a slit from two slit sources which are separated by a distance such that their separation subtends an angle θ at the slit. Then each slit source will give rise to its own diffraction pattern, and the central maxima of the patterns will be separated by an angle θ.

To decide whether we can distinguish between two diffraction patterns, we use a (somewhat arbitrary) criterion due to Lord Rayleigh:

two peaks in an interference pattern can be distinguished if the central maximum of one peak falls on or beyond the first minimum of the other.

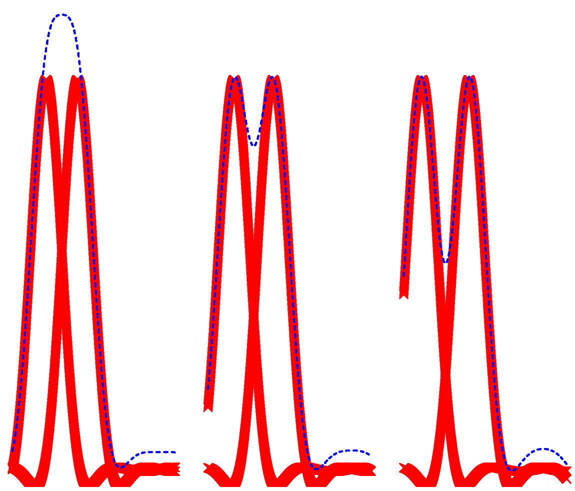

Figure L25.1: Rayleigh's criterion for the distinguishability of two diffraction peaks, illustrated by peaks separated by 0.8, 1.0 and 1.2 times the Rayleigh distance.

As figure L25.1 shows, this rule does give a perceptible dip in intensity between the two peaks. If we apply Rayleigh's criterion, this tells us that the diffraction patterns of the slits will only be distinguished if

$$\theta \geq \frac{\lambda}{d}.$$

L25.2 Resolution of a circular aperture

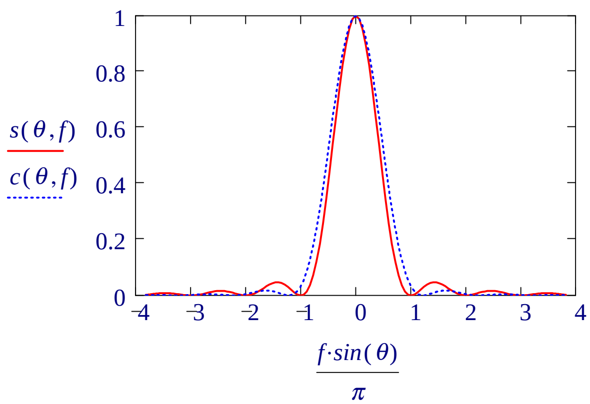

Figure L25.2: The diffraction pattern $c(f, \theta)$ of a circular aperture compared with $s(f, \theta)$ for a slit, where f is $\pi d/\lambda$. The wavelength is λ and d is the width of the slit or the diameter of the circular aperture.

The pattern produced by a circular aperture is described in terms of special functions known as Bessel functions[1]. The diffraction pattern is a little wider than that of the slit, as shown in figure L25.2, and for our purposes the important result is that

[1]Basically, Bessel functions play the same role in terms of standing waves in circular cavities as sine and cosine functions do for one-dimensional systems and for rectangular and cuboidal cavities. They are oscillatory functions, but decrease in amplitude with distance from the origin, in line with our earlier conclusion that energy conservation required waves from a line source should fall off as the square root of the distance.

the diffraction patterns of two sources with an angular separation viewed through a circular aperture of diameter d at wavelength λ can be distinguished if

$$\theta \geq 1.22\frac{\lambda}{d}.$$

This limits the resolution of all optical devices, ranging from telescopes to the eyes of living creatures.

Abbe theory *Hecht562-564*

The discussion of resolution above looked at the overlap of diffraction patterns in terms of their intensity patterns only. This involves an implicit assumption that the two light sources are not mutually coherent, so that there is no interference pattern. When we look at microscopes, however, it is quite likely that signals from nearby points will be coherent. To study this case, consider forming the image of a diffraction grating. We know that in the far field of a grating with spacing h illuminated by a plane coherent wave of wavelength λ the three maxima of the diffraction pattern closest to the axis are the straight-through signal and the first-order diffraction peaks at angles $\pm\theta$ where

$$\sin(\theta) = \frac{\lambda}{h}.$$

Without the first-order peaks, there is *no* information about the grating spacing: in other words, if the light from the grating passes through an aperture the spatial information will be lost if the aperture subtends an angle less than 2θ at the grating (see figure L25.3). In the case of a microscope, the aperture is often the diameter of the obejctive lens. Turning this round,

if the angle subtended at the object by the objective lens is $2\theta_o$, the finest scale that can be resolved with a wavelength λ is

$$h_{\min} = \frac{\lambda}{\sin(\theta_o)}.$$

It may be noted that an argument based on Rayleigh's criterion gives rise to the very similar criterion

$$h_{\min} = \frac{1.22\lambda}{2\sin(\theta_o)}.$$

A little thought shows that the finite aperture acts as a *spatial filter*, as it is the larger angles, corresponding to smaller spacings, which are cut out by the aperture.

L25.3 Fraunhofer diffraction *AF939-945*

Now that we know about the diffraction pattern of a slit, and indeed about the general expression for the diffraction from a screen with any pattern, we can see what happens if we include the sizes of the slits in the grating pattern.

In our previous treatment we included a term (E_1) which represented the angular pattern of each element of the grating. At that point we assumed that each element was infinitesimally narrow, so each slit radiated uniformly in all directions. Suppose, now, that those elements are replaced by identical slits: keep the slit spacing at h, but replace E_1 with the field pattern for a slit of width d. Then the overall intensity will be

$$\boxed{I(\theta) = I(0)\left[\frac{\sin\left(\frac{kd\sin(\theta)}{2}\right)}{\frac{kd\sin(\theta)}{2}}\right]^2\left[\frac{\sin\left(\frac{Nkh\sin(\theta)}{2}\right)}{N\sin\left(\frac{kh\sin(\theta)}{2}\right)}\right]^2.}$$

We will give here a slightly more detailed treatment which shows that the factorisation, which was implicit in that way of writing the result, is correct. Note that this derivation was not given in the lecture: it is included here for completeness, but you need not learn it.

grating *FGT1051-1055, AF945*

The grating is a pattern of N slits, each of width d, and spaced at a distance h between their centres along the y axis, so that we may write the total field at a point a distance D in front of the screen at an angle θ to the axis in the form

$$\begin{aligned} E = \quad & C \quad e^{i(\omega t - kD)}\left[\int_{-d/2}^{d/2} e^{-iky\sin(\theta)}\mathrm{d}y\right. \\ & + \int_{h-d/2}^{h+d/2} e^{-iky\sin(\theta)}\mathrm{d}y \\ & + \int_{2h-d/2}^{2h+d/2} e^{-iky\sin(\theta)}\mathrm{d}y + \ldots \end{aligned}$$

$$+ \quad \int_{(N-1)h-d/2}^{(N-1)h+d/2} e^{-iky\sin(\theta)}\mathrm{d}y \Bigg] .$$

Now each integral is of the form

$$\begin{aligned}\int_{nh-d/2}^{nh+d/2} e^{-iky\sin(\theta)}\mathrm{d}y &= \left[\frac{e^{-ikx\sin(\theta)}}{ik\sin(\theta)}\right]_{nh-d/2}^{nh+d/2} \\ &= e^{-iknh\sin(\theta)}\left[\frac{2\sin\left(\frac{kd\sin(\theta)}{2}\right)}{k\sin(\theta)}\right]\end{aligned}$$

that is, there is a common factor

$$\left[\frac{\sin\left(\frac{kd\sin(\theta)}{2}\right)}{\frac{k\sin(\theta)}{2}}\right],$$

which is the diffraction pattern of each slit (our previous E_1), and a geometric series with typical term

$$e^{-iknh\sin(\theta)}$$

i.e. a common ratio

$$e^{-ikh\sin(\theta)}$$

which we have dealt with before. We can sum this geometric series to a N terms.

Overall, then, the diffracted amplitude from the grating is the product of the pattern from the arrangement of slits and the pattern from each slit (as we assumed before), and the intensity has the form

$$I(\theta) = I(0)\left[\frac{\sin\left(\frac{kd\sin(\theta)}{2}\right)}{\frac{kd\sin(\theta)}{2}}\right]^2\left[\frac{\sin\left(\frac{Nkh\sin(\theta)}{2}\right)}{N\sin\left(\frac{kh\sin(\theta)}{2}\right)}\right]^2 .$$

finite size of grating ***FGT1054***

This shows that when the finite slit size is included, the pattern of principal maxima and minima, and the dependence of the sharpness on the pattern on the number of slits, are unaffected.

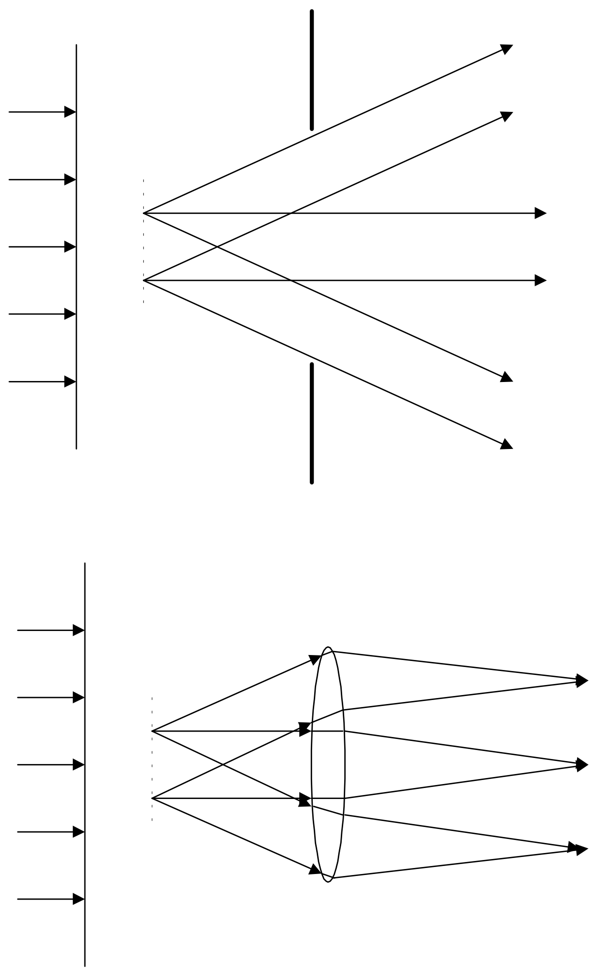

Figure L25.3: The lowest-order diffraction maxima of a grating passing through a finite aperture, such as the objective lens of a mircoscope. The upper diagram shows a simple aperture, whereas the lower diagram shows the focusing effect of the lens.