Topic 18 — Standing Waves and Waveguides

Note that this topic is presented only as background material and is not material which will be asked about in the examination.

Just as in one dimension (standing waves on a string or on a bar) standing waves exist in two and three dimensions. Again, it is the combination of the differential equation (the wave equation) and the boundary conditions which give the standing waves.

T18.1 Standing waves *AF919*

Standing waves in two and three dimensions *AF926- 929*

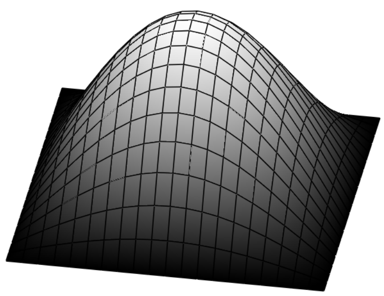

Figure T18.1: The lowest mode of vibration of a square membrane.

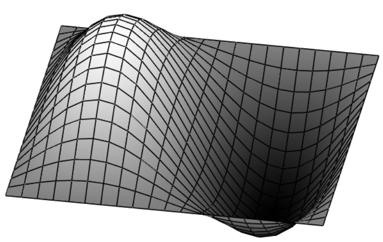

Figure T18.2: A second mode of vibration of a square membrane.

The simplest case is a two-dimensional square drum with sides L_x and L_y: the movement of the drum membrane may be written as a wave

$$\xi(x, y, t) = ae^{i(\omega t - k_x x - k_y y)}$$

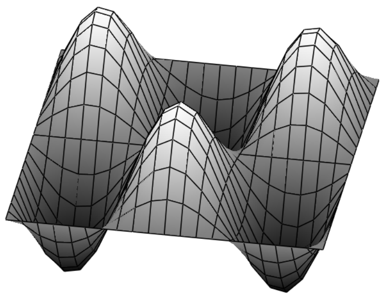

Figure T18.3: A third mode of vibration of a square membrane.

which reduces with fixed boundary conditions to a product of sine waves for each direction of the form

$$\xi(x, y, t) = a' e^{i\omega t} \sin\left(\frac{n_x \pi x}{L_x}\right) \sin\left(\frac{n_y \pi y}{L_y}\right)$$

where n_x and n_y are integers. Typical standing wave patterns on a square drum are shown in figures T18.1, T18.2 and T18.3.

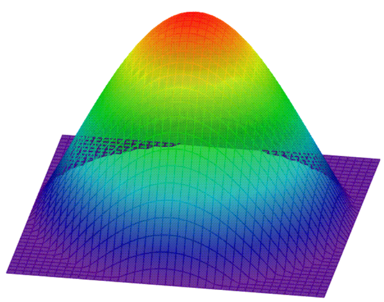

Figure T18.4: The lowest mode of vibration of a circular membrane.

For a circle, the displacement again factors, but this time into a product of radial and angular functions. The radial functions include oscillations, but also have an amplitude which varies with distance from the origin[1]. Two of the lower-frequency modes are shown in figures T18.4 and T18.5.

[1]This amplitude variation is in line with the $1/\sqrt{r}$ variation that we predicted on energy grounds in the last lecture. Technically, the functions involved are known as Bessel functions, and are written as $J_n(kr)$ for integer n.

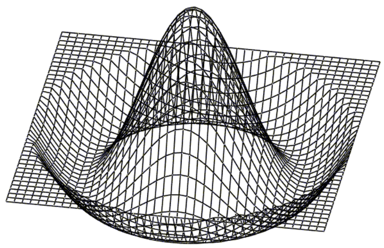

Figure T18.5: A mode of vibration of a circular membrane, showing one radial node but maintaining circular symmetry.

T18.2 sound waves in pipes - end corrections *AF922*

If the boundary conditions do not correspond to rigid edges, we have to be a little more careful. Consider a pipe that is open at one end. Why should the pressure in the wave suddenly drop to zero there? Surely the sound wave will 'leak' from the end? Yes, it will – the maths becomes complicated, but in general a good approximation is to add an 'end correction' of the order of 0.8 times the radius of the tube.

T18.3 Simple waveguides *AF930-933*

There are many situations in which we want to generate a wave at one point and send it to another point rather than broadcasting it over a wide area. That is, we want to 'pipe' waves along specific paths. The full theory of the waveguides which can accomplish this is rather complicated, but there is a half-way house in terms of standing waves in two or three dimensions, in which a wave is confined between two surfaces and guided in a particular direction. This simple case illustrates several of the important features of waveguides.

Suppose we have a wave in two dimensions $\xi(x, y, t)$ confined between planes at $y = 0$ and $y = a$, and treat the planes as rigid boundaries (figure T18.6. A wave that starts off with wave-vector $\mathbf{k} = (k_x, k_y)$ will be reflected off the top surface, as a result of which its k_y component will be reversed but its k_x component will be left alone. Its next reflection will reverse k_y back to its original value. At any point in the waveguide, then,

$$\xi(x, y, t) = Ae^{i(\omega t - k_x x - k_y y)} + Be^{i(\omega t - k_x x + k_y y)}$$

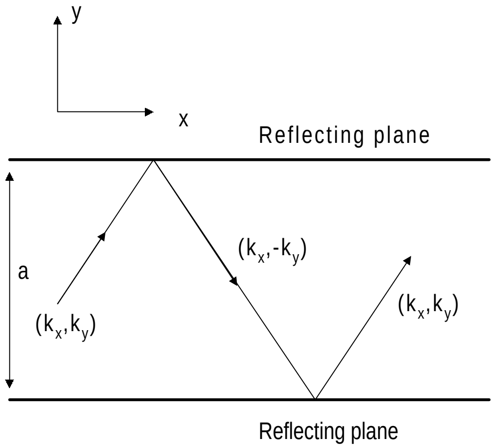

Figure T18.6: A pair of reflecting planes, forming a wave-guide.

with $\xi = 0$ at $y = 0$, $y = a$ so

$$\xi(x, y, t) = C \sin(k_y y) e^{i(\omega t - k_x x)}$$

with $k_y = \frac{n\pi}{a}$.

That is, we have a standing wave in the y direction but a travelling wave in the x direction. The amplitude will have nodal planes perpendicular to the y direction, where the sine function is zero.

The phase velocity in the x direction is

$$v_p = \frac{\omega}{k_x}.$$

But if the wave velocity in free space is c, the components k_x and k_y satisfy

$$k_x^2 + k_y^2 = \frac{\omega^2}{c^2}.$$

We know the form of k_y, so

$$\begin{aligned} k_x^2 &= \frac{\omega^2}{c^2} - \frac{n^2\pi^2}{a^2} \\ &= \left(\frac{\omega}{c}\right)^2 \left(1 - \frac{n^2\pi^2 c^2}{\omega^2 a^2}\right). \end{aligned}$$

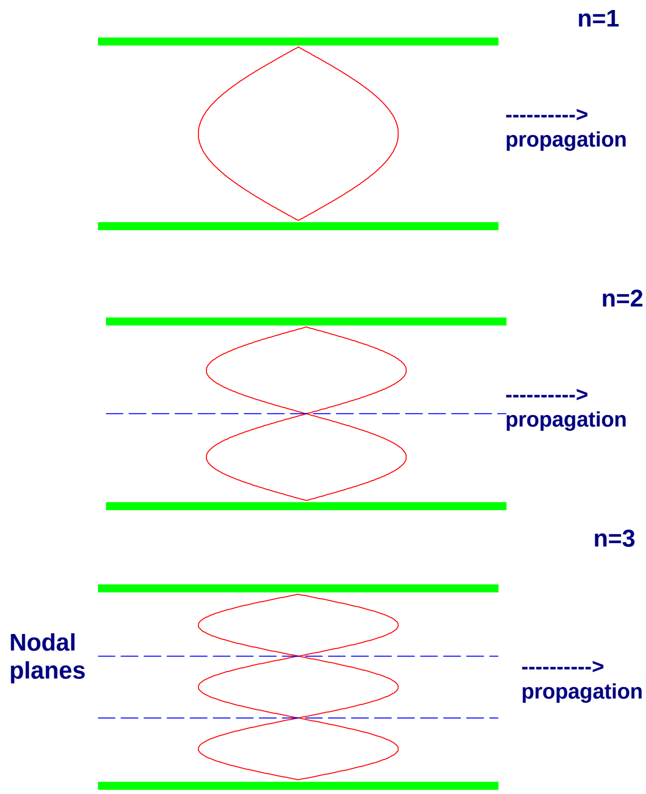

Figure T18.7: The nodal planes in a two-dimensional wave guide.

From this we can obtain the phase velocity, which is the velocity in the direction in which the wave is actually propagating,

$$\begin{aligned} v_p &= \frac{\omega}{k_x} \\ &= \frac{\omega}{\sqrt{\frac{\omega^2}{c^2} - \frac{n^2\pi^2}{a^2}}} \\ &= \frac{c}{\sqrt{1 - \frac{n^2\pi^2c^2}{\omega^2a^2}}} \end{aligned}$$

i.e. the phase velocity is *greater* than the free space velocity.

The group velocity, on the other hand, is

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k_x}$$

so

$$2k_x = \frac{2\omega}{c^2}\frac{\mathrm{d}\omega}{\mathrm{d}k_x}$$

or

$$v_g = c^2\frac{k_x}{\omega} = \frac{c^2}{v_p}$$

so the group velocity is less than c.

Note that if

$$\frac{n^2\pi^2c^2}{\omega^2a^2} < 1$$

k_x and the velocity become *imaginary*. But imaginary k_x converts the complex exponential into a real one – the signal *decays* with distance into the guide. This means that there is a *lower cutoff-frequency* which is different for each *mode* or value of n.

Optical fibres are examples of guided-wave systems, for light. Acoustic waveguides are common – the inner ear is an example – and their potential for eavesdropping is famous[2].

[2] *Acoustics, Musurgia Universalis* is a fascinating book written in 1650 by Athanasius Kircher, a Jesuit priest. The book includes illustrations of a building with horns shaped like conch shells embedded in the walls to gather sound and focus it at secret listening posts.