# Topic 1 – Simple Oscillations and their Description

Waves play a crucial role in an enormous range of physical phenomena, and are involved in most of our everyday experiences. The light by which you can see me, the sound of my speaking, are examples of waves. Modern communication, by radio, microwave, or optic fibre, exploits the properties of waves. Down in the fundamental structure of matter it is wave-like properties of particles that control the quantum world, and which in turn determines the way everyday material behaves. The aim of this course is to teach you about the properties of waves in general, and some aspects of light and sound in particular. We will learn when we have to worry about the wave nature of light, and when it is sufficient to think of light travelling in straight lines.

Before we discuss waves, though, we need to establish some basic ideas and notations. For these we will have to remind ourselves of the behaviour of simple harmonic oscillators – in a sense the first few lectures might be called 'all dressed up with nowhere to go'. The link to waves will be developed gradually, after we have looked at some general features of waves, as we move from the properties of isolated oscillators through a chain of masses linked by springs. Figures 0.1 and 0.2 show a range of oscillatory systems.

## T1.1 Simple Harmonic Oscillations *FGT345-374, AF194, AF197-198*

### frictionless mass on spring *FGT350, AF194*

The prototypical simple harmonic oscillator is a mass on spring on a flat frictionless surface (to avoid gravity, even though gravity won't affect the result). We assume a linear spring, with a force $F = -kx$ opposing a stretch of $x$.

$$m\frac{d^2x}{dt^2} = -kx. \tag{0.1}$$

Important point: *linear* equation, so if $x_1$ and $x_2$ are solutions, so is $x_1 + x_2$. Check:

$$m\frac{d^2x_1(t)}{dt^2} = -kx_1(t) \tag{0.2}$$

$$m\frac{d^2x_2(t)}{dt^2} = -kx_2(t) \tag{0.3}$$

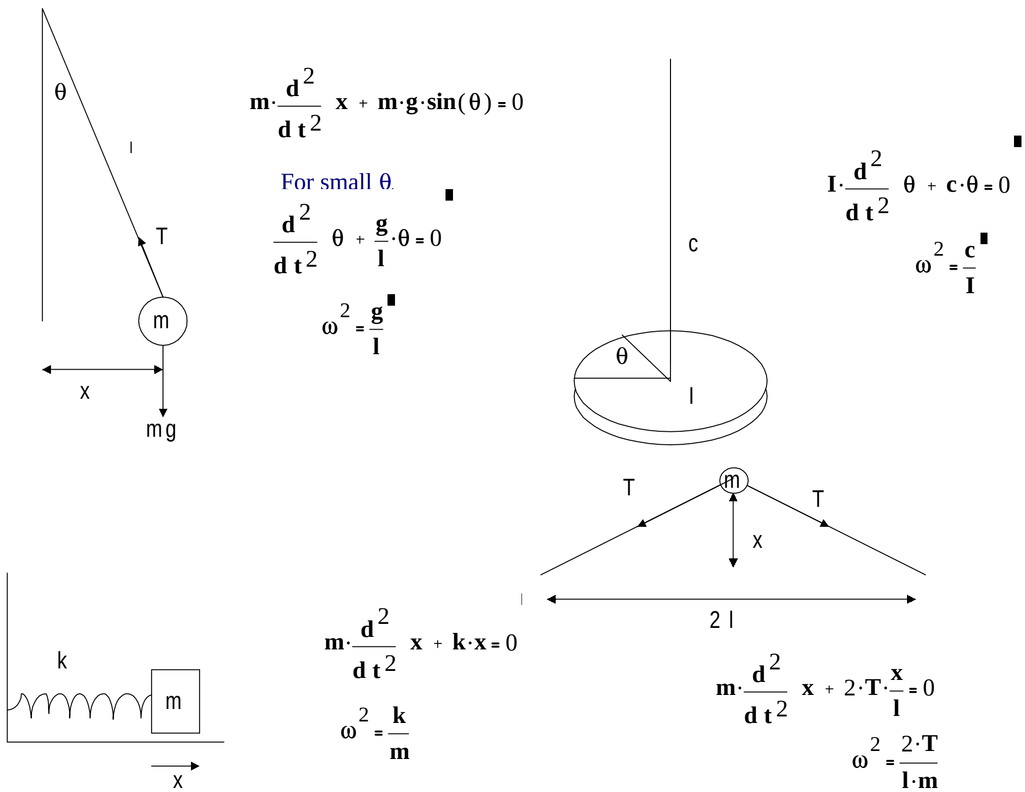



Figure 0.1: A selection of simple harmonic oscillators.

$$m\frac{d^2(x_1(t)+x_2(t))}{dt^2} = -k(x_1(t)+x_2(t)). \tag{0.4}$$

$$\tag{0.5}$$

Similarly, if $x(t)$ is a solution, any constant multiple of $x(t)$ is a solution. Note that if equation had been, for example,

$$m\frac{d^2x}{dt^2} = -kx^2$$

then this would not be true.

Now we know that we can satisfy equation 0.1 with either $x(t) = \sin(\omega_0 t)$ or $x(t) = \cos(\omega_0 t)$ provided that $\omega_0 = \sqrt{k/m}$. This is the characteristic motion of a simple harmonic oscillator. $\omega_0$ is the angular frequency of the oscillation. We know that each of these functions starts to repeat itself after its argument $\omega_0 t$ reaches $2\pi$, and the corresponding time $t = 2\pi/\omega_0$ is the period of the wave, $\tau$.

Note that the form of the frequency is characteristic of a lot of waves: it is the square root of some sort of modulus, or stiffness, divided by a measure of inertia. The heavier the mass the more sluggish the motion and the longer the period.

### General Solution

To find the general solution of the SHO equation we may need the cos, the sin, or a combination. cos alone corresponds to starting the motion with a maximum displacement at $t = 0$, and so far we have not specified the magnitude of the displacement. sin alone corresponds to starting with no displacement, but of course with a non-zero velocity $v = (d/dt)\sin(\omega_0 t)$. The general solution is

$$x(t) = a\cos(\omega_0 t) + b\sin(\omega_0 t).$$

With no less generality we can write

$$\begin{aligned} a &= A\cos(\phi) \\ b &= -A\sin(\phi) \end{aligned}$$

(which still allows $a$ and $b$ independently to take any value between plus and minus infinity), and then

$$x(t) = A\cos(\phi)\cos(\omega_0 t) - A\sin(\phi)\sin(\omega_0 t) = A\cos(\omega_0 t + \phi).$$

The maximum displacement is $A$, the amplitude of the oscillation, and $\phi$ is the phase shift. This is, of course, just equivalent to a shift in time, as we can write $\phi = \omega_0 t_0 \omega$ so that

$$x(t) = A\cos(\omega_0(t + t_0)).$$

### Energy

At any instant, the system of mass and spring typically has kinetic and potential energy. The kinetic energy, $\frac{1}{2}mv^2$, is often denoted by $T$, and is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}mA^2\omega_0^2\sin^2(\omega_0(t + t_0)).$$

The potential energy $U$ is $\frac{1}{2}kx^2$

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega_0(t+t_0)).$$

This shows us three things:

- first, the kinetic and potential energies are always positive;
- second, energy is exchanged between kinetic and potential;
- third, the total energy (remembering that $k = m\omega_0^2$) is

  $$E = T+U = \frac{1}{2}m\omega_0^2A^2\left[\sin^2(\omega_0(t+t_0)) + \cos^2(\omega_0(t+t_0))\right] = \frac{1}{2}m\omega_0^2A^2,$$

  which is constant.

Of course, in general there will be some loss in the system (for example, heating of the spring as a result of the repeated compression and extension), but we will ignore that for the present.

**pendulum (derivation not required)** ***AF197- 198***

Other oscillating systems can be described in just the same way. The classic example is the pendulum:

$$\begin{aligned} m\ddot{x} + mg\sin\theta &= 0 \\ ml\ddot{\theta} + mg\theta &= 0 \\ \omega_0^2 &= \frac{g}{l}. \end{aligned}$$

## T1.2 Motion in a circle, projection on a line, phasors *FGT915-916, AF193*

Sine and cosine are known as circular functions[1], and there is a close link between the simple harmonic oscillator and motion in a circle. Imagine a disk of radius $A$, turning around its axis with angular frequency $\omega_0$. If a point on

[1]Because as functions of a continuous parameter they trace the projections on the axes of a point moving around a circle. You might like to consider why cosh and sinh are known as hyperbolic functions.

the circumference lies on the positive $x$ axis at time 0, then the radius vector to that point on the circumference will have turned to an angle $\theta$ in a time $\theta/\omega_0$. In fact its coordinates in the xy plane will be $(A\cos(\theta), A\sin(\theta))$, or $(A\cos(\omega_0 t), A\sin(\omega_0 t)$.

The rotating vector is called a **phasor**.

If we imagine a light pinned to the wheel, if we look along the axis of the wheel we will see the light describing a circle. If we look down at the edge of the wheel we will see the light going up-down-up-... . But it's making that movement in a rather special way. What we see is just the $y$ position of the light, which is $A\cos(\omega_0 t)$ - in other words it is executing simple harmonic motion:

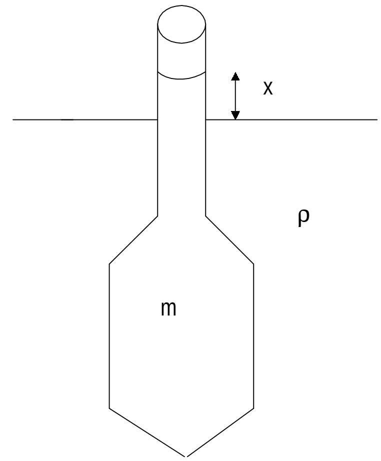
simple harmonic motion is the projection of circular motion onto one axis.

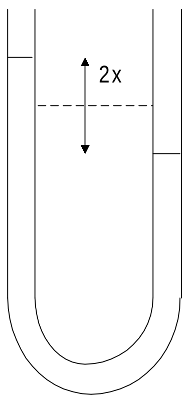
## T1.3 Maths used in this Topic

### Trigonometric Formulae

$$\cos(a)\cos(b) - \sin(a)\sin(b) = \cos(a+b)$$

$$\cos^2(a) + \sin^2(a) = 1$$




$$\rho \cdot l \cdot \frac{d^2}{dt^2} x + 2 \cdot \rho \cdot g \cdot x = 0$$

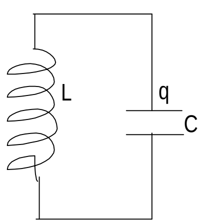
where l is the length of the fluid column

$$\omega^2 = 2 \cdot \frac{g}{l}$$

$$m \cdot \frac{d^2}{dt^2} x + A \cdot \rho \cdot g \cdot x = 0$$

where A is the cross-sectional area of the tube

$$\omega^2 = \frac{A \cdot \rho \cdot g}{m}$$




$$L \cdot \frac{d^2}{dt^2} q + \frac{q}{C} = 0$$

$$\omega^2 = \frac{1}{L \cdot C}$$

Figure 0.2: A further selection of simple harmonic oscillators.

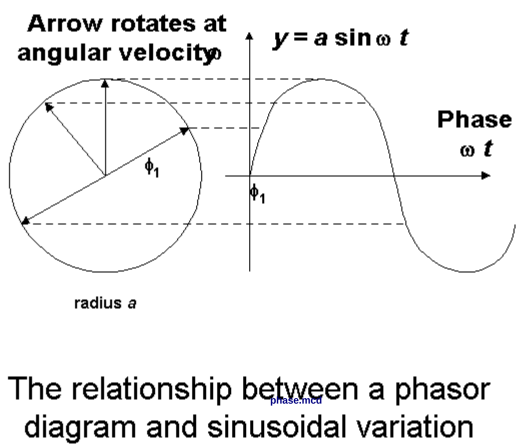



Figure 0.3: Relationship between a rotating vector and simple harmonic motion.