Topic 31 — Applications of Refraction I

L31.1 Approximations and Sign Conventions

Approximations

Throughout the treatment of optical instruments we make the *paraxial approximation*, that the rays make a small enough angle α to the axis that

$$\sin(\alpha) \approx \tan(\alpha) \approx \alpha,$$

although most of the figures we draw will exaggerate the angles in order to make the angles visible. These small-angle rays are known as *paraxial rays*.

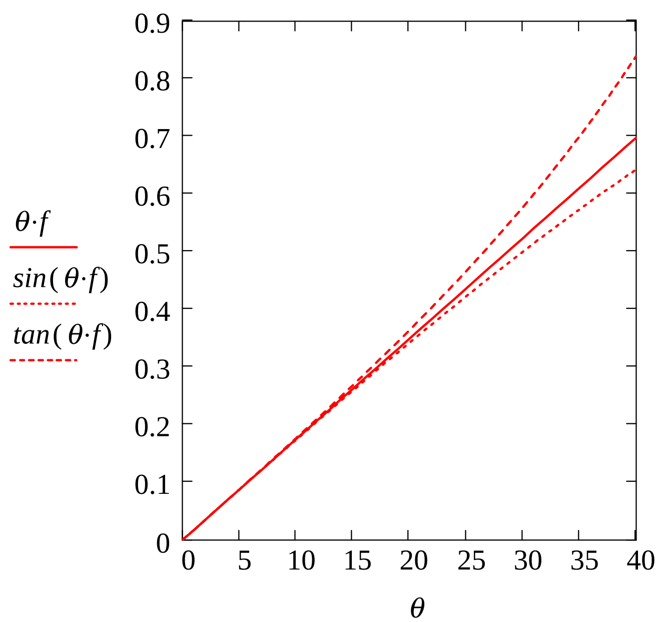

Figure L31.1: Illustration of the approximation $\theta \approx \sin(\theta) \approx \tan(\theta)$ for small angles. Note that although θ on the horizontal axis is given in degrees, on the vertical axis the factor $f = \pi/180$ converts θ to radians.

A simple graph, as in figure L31.1 shows that this approximation is valid over quite a large range of angles.

Sign Conventions

In order to get unambiguous formulae, we need to adopt a sign convention.

We choose a so-called Cartesian convention, in which

- we define an *optical axis* which passes through the centre of each element (mirror, lens, curved interface) and through its centre of curvature (thus the axis intersects each surface perpendicularly);
- the origin of the Cartesian system is located at the vertex of the curved boundary or mirror, and at the centre of a thin lens, with the x axis directed along the optical axis from left to right;
- object, image, and centre of curvature distances are defined to be the x coordinates of the y, z planes which contain them (thus distances to points or planes to the right of the vertex or lens centre are positive, those to the left are negative);
- light sources and objects are placed to the left of the first surface in the system, so that the light rays travel from left to right, but the object has a negative x coordinate and the object distance will thus be negative;
- angles are taken to be positive or negative dependent on whether their tangents are positive or negative;

The way we use these sign conventions is two-fold

- when *deriving* equations, just use geometry and put in the signs at the end to get general-purpose equations;
- when *using* the general-purpose equations, substitute distances *with the appropriate signs.*

[1]

[1] Note that we shall use the Cartesian sign convention, that is Group I case 1 of T. Smith's report on the Teaching of Geometrical Optics (1934). This appears to be the one in most common use in more advanced books on optics. One user of an alternative (Group II case 1) scheme, in which distances are counted positive if they are actually traversed by a light beam, negative if not (i.e. if they lead to a virtual image) is S.C. Strong Concepts of Classical Optics, Freeman (1958). Note that Hecht uses a virtual image negative convention, as do Alonso and Finn and Fishbane, Gasiorowicz and Thornton.

L31.2 Lenses ***AF882-889***

refraction at convex spherical surfaces *AF882- 884*

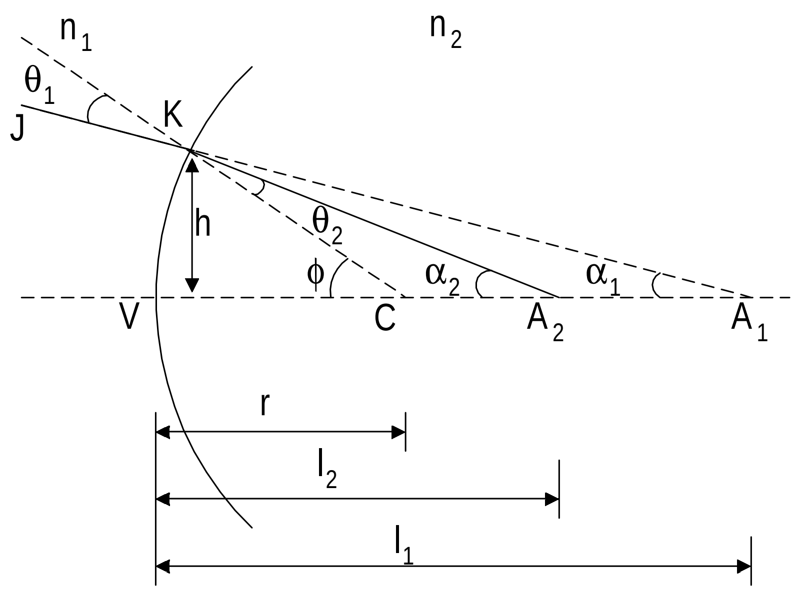

Figure L31.2: Refraction of light at a convex spherical interface.

We treat light incident from the left (in accordance with our conventions) on a curved interface between the incident medium with refractive index n_1 and a medium with refractive index n_2. In figure L31.2 the solid line JKA_2 shows the ray, with its refraction at the interface at K.

From triangle KCA_1 we have (using the fact that the external angle of a triangle is equal to the sum of the two opposite internal angles)

$$\phi = \alpha_1 + \theta_1$$

and from triangle KCA_2

$$\phi = \alpha_2 + \theta_2.$$

But Snel's law tells us that

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2),$$

and assuming that the angles are small, we may write this as

$$n_1\theta_1 = n_2\theta_2.$$

Thus

$$n_1(\phi - \alpha_1) = n_2(\phi - \alpha_2)$$

and we may write this as

$$n_1\left(\frac{h}{r} - \frac{h}{l_1}\right) = n_2\left(\frac{h}{r} - \frac{h}{l_2}\right)$$

which may be tidied up by cancelling h to

$$\boxed{\frac{n_1}{l_1} - \frac{n_2}{l_2} = \frac{n_1 - n_2}{r}.}$$

We now need to insert the signs appropriate to our sign convention. As all the positions are at positive x, no change is necessary.

Note that this only holds for small angles, i.e. in the so-called *paraxial approximation*.

refraction at concave spherical surfaces *AF882- 884*

We can do the corresponding calculation for a concave surface, as shown in figure L31.3. In this case the angles of incidence and refraction are the external angles of the relevant triangles, and we have from triangle KCA_1

$$\theta_1 = \phi + \alpha_1$$

and from KCA_2

$$\theta_2 = \phi + \alpha_2$$

and turning these round and using Snel's law again for small angles gives

$$n_1(\phi + \alpha_1) = n_2(\phi + \alpha_2)$$

which leads to

$$\frac{n_1}{r} + \frac{n_1}{l_1} = \frac{n_2}{r} + \frac{n_2}{l_2}$$

or

$$\frac{n_1}{l_1} - \frac{n_2}{l_2} = \frac{n_2 - n_1}{r}$$

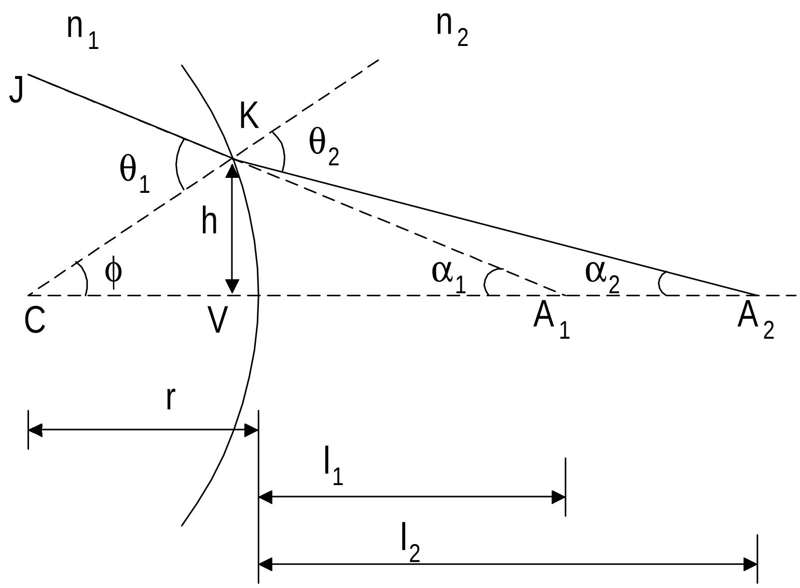

Figure L31.3: Refraction of light at a concave spherical interface.

which, at first sight, looks different from the result for the convex surface. But our sign convention tells us that the position of C is a negative value of x, so we recover

$$\boxed{\frac{n_1}{l_1} - \frac{n_2}{l_2} = \frac{n_1 - n_2}{r}}$$

and we can state that this equation works for either convex or concave surfaces, provided we use the sign convention.

We can find the positions of the foci. A focus corresponds to the source position which will give rise to an image at infinity (rays leaving the system parallel to the optical axis) or to the point at which rays from a source at infinity (incoming rays parallel to the optical axis) cross the axis.

The first focal point corresponds to putting $l_2 = \infty$ so that

$$\frac{1}{f_1} = \frac{n_1 - n_2}{n_1}\frac{1}{r}$$

and the other focal point is at

$$\frac{1}{f_2} = \frac{n_2 - n_1}{n_2}\frac{1}{r}.$$

image formation by spherical surfaces *AF884- 885*

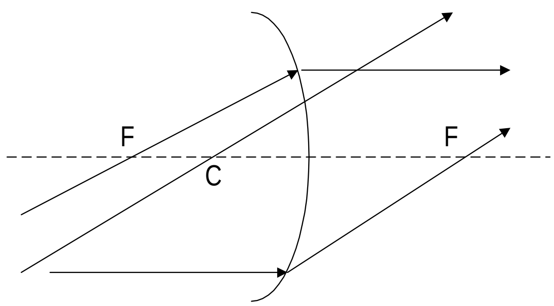

Figure L31.4: The principal rays at a curved interface: these are the two rays through the foci, which enter or leave the system parallel to the optical axis, and the one ray which passes through the system undeviated (in this case, the ray through the centre of curvature of the interface).

For the refracting surface we can now draw the *principal rays*. These rays are as follows (see figure L31.4):

- one ray which comes through a point on the optical axis (the first principal point or object focus) and leaves travelling the left parallel to the principal axis,
- one ray which comes in from the left parallel to the principal axis, which will be refracted through the second principal point or the image focus,
- one ray which passes through the interface undeviated.

For this surface, we can now study the formation of an image by using the principal rays. The easiest way is to draw an object with its base on the principal axis, use the formulae to find the position of the image, and then draw the principal rays.

magnification by spherical surfaces *AF884- 885*

Figure L31.5: Image formation at a spherical interface, showing the object with a height h_1 and the image with height h_2.

Figure L31.5 shows the formation of an image in a spherical interface. The magnification is the ratio of the image height to the object height:

$$M = \frac{h_2}{h_1} = \frac{\theta_2\ \overline{\mathrm{A}_2\mathrm{V}}}{\theta_1\overline{\mathrm{A}_1\mathrm{V}}} = \frac{n_1\ \overline{\mathrm{A}_2\mathrm{V}}}{n_2\overline{\mathrm{A}_1\mathrm{V}}} = \frac{n_1\ l_2}{n_2\ l_1}.$$

As an example, consider the magnification produced by a single refracting surface. Take a concave surface, radius of curvature 0.5 m, between a medium of refractive index 1.2 and one with refractive index 1.6. Place an object in the first medium, 0.8 m from the interface.

From the diagram, we have here $r = -0.5m$, $l_1 = -0.8m$, and so

$$-\frac{1.2}{0.8} - \frac{1.6}{l_2} = -\frac{1.2 - 1.6}{0.5}$$

from which we deduce that the image is at $l_2 = -0.69m$, which is on the same side of the surface as the object. This, then, will be a virtual image (the rays themselves do not form the image, the rays produced do).

The magnification will be

$$M = \frac{1.2 \times 0.69}{1.6 \times 0.80} = 0.65$$

which is positive but less than 1, giving an erect but reduced image.

The focal points are at

$$-r\frac{n_2}{n_1 - n_2} = 0.5\frac{1.6}{0.4} = 2.0$$

and

$$r\frac{n_1}{n_1 - n_2} = 0.5\frac{1.2}{-0.4} = -1.5.$$