# PART 1: INTRODUCTION TO TENSOR CALCULUS

A scalar field describes a one-to-one correspondence between a single scalar number and a point. An n-dimensional vector field is described by a one-to-one correspondence between n-numbers and a point. Let us generalize these concepts by assigning $n$-squared numbers to a single point or $n$-cubed numbers to a single point. When these numbers obey certain transformation laws they become examples of tensor fields. In general, scalar fields are referred to as tensor fields of rank or order zero whereas vector fields are called tensor fields of rank or order one.

Closely associated with tensor calculus is the indicial or index notation. In section 1 the indicial notation is defined and illustrated. We also define and investigate scalar, vector and tensor fields when they are subjected to various coordinate transformations. It turns out that tensors have certain properties which are independent of the coordinate system used to describe the tensor. Because of these useful properties, we can use tensors to represent various fundamental laws occurring in physics, engineering, science and mathematics. These representations are extremely useful as they are independent of the coordinate systems considered.

## §1.1 INDEX NOTATION

Two vectors $\vec{A}$ and $\vec{B}$ can be expressed in the component form

$$\vec{A} = A_1\,\widehat{\mathbf{e}}_1 + A_2\,\widehat{\mathbf{e}}_2 + A_3\,\widehat{\mathbf{e}}_3 \qquad \text{and} \qquad \vec{B} = B_1\,\widehat{\mathbf{e}}_1 + B_2\,\widehat{\mathbf{e}}_2 + B_3\,\widehat{\mathbf{e}}_3,$$

where $\widehat{\mathbf{e}}_1$, $\widehat{\mathbf{e}}_2$ and $\widehat{\mathbf{e}}_3$ are orthogonal unit basis vectors. Often when no confusion arises, the vectors $\vec{A}$ and $\vec{B}$ are expressed for brevity sake as number triples. For example, we can write

$$\vec{A} = (A_1,\ A_2,\ A_3) \qquad \text{and} \qquad \vec{B} = (B_1,\ B_2,\ B_3)$$

where it is understood that only the components of the vectors $\vec{A}$ and $\vec{B}$ are given. The unit vectors would be represented

$$\widehat{\mathbf{e}}_1 = (1,0,0), \qquad \widehat{\mathbf{e}}_2 = (0,1,0), \qquad \widehat{\mathbf{e}}_3 = (0,0,1).$$

A still shorter notation, depicting the vectors $\vec{A}$ and $\vec{B}$ is the index or indicial notation. In the index notation, the quantities

$$A_i, \quad i = 1,2,3 \qquad \text{and} \qquad B_p, \quad p = 1,2,3$$

represent the components of the vectors $\vec{A}$ and $\vec{B}$. This notation focuses attention only on the components of the vectors and employs a dummy subscript whose range over the integers is specified. The symbol $A_i$ refers to all of the components of the vector $\vec{A}$ simultaneously. The dummy subscript $i$ can have any of the integer values $1, 2$ or 3. For $i = 1$ we focus attention on the $A_1$ component of the vector $\vec{A}$. Setting $i = 2$ focuses attention on the second component $A_2$ of the vector $\vec{A}$ and similarly when $i = 3$ we can focus attention on the third component of $\vec{A}$. The subscript $i$ is a dummy subscript and may be replaced by another letter, say $p$, so long as one specifies the integer values that this dummy subscript can have.

It is also convenient at this time to mention that higher dimensional vectors may be defined as ordered $n-$tuples. For example, the vector

$$\vec{X} = (X_1, X_2, \dots, X_N)$$

with components $X_i,\ i = 1, 2, \dots, N$ is called a $N-$dimensional vector. Another notation used to represent this vector is

$$\vec{X} = X_1\,\widehat{\mathbf{e}}_1 + X_2\,\widehat{\mathbf{e}}_2 + \cdots + X_N\,\widehat{\mathbf{e}}_N$$

where

$$\widehat{\mathbf{e}}_1,\ \ \widehat{\mathbf{e}}_2, \dots,\ \widehat{\mathbf{e}}_N$$

are linearly independent unit base vectors. Note that many of the operations that occur in the use of the index notation apply not only for three dimensional vectors, but also for $N-$dimensional vectors.

In future sections it is necessary to define quantities which can be represented by a letter with subscripts or superscripts attached. Such quantities are referred to as systems. When these quantities obey certain transformation laws they are referred to as tensor systems. For example, quantities like

$$A^k_{ij} \qquad e^{ijk} \qquad \delta_{ij} \qquad \delta^j_i \qquad A^i \qquad B_j \qquad a_{ij}.$$

The subscripts or superscripts are referred to as indices or suffixes. When such quantities arise, the indices must conform to the following rules:

1. They are lower case Latin or Greek letters.
2. The letters at the end of the alphabet $(u, v, w, x, y, z)$ are never employed as indices.

The number of subscripts and superscripts determines the order of the system. A system with one index is a first order system. A system with two indices is called a second order system. In general, a system with $N$ indices is called a $N$th order system. A system with no indices is called a scalar or zeroth order system.

The type of system depends upon the number of subscripts or superscripts occurring in an expression. For example, $A^i_{jk}$ and $B^m_{st}$, (all indices range 1 to N), are of the same type because they have the same number of subscripts and superscripts. In contrast, the systems $A^i_{jk}$ and $C^{mn}_p$ are not of the same type because one system has two superscripts and the other system has only one superscript. For certain systems the number of subscripts and superscripts is important. In other systems it is not of importance. The meaning and importance attached to sub- and superscripts will be addressed later in this section.

In the use of superscripts one must not confuse "powers "of a quantity with the superscripts. For example, if we replace the independent variables $(x, y, z)$ by the symbols $(x^1,\ x^2,\ x^3)$, then we are letting $y = x^2$ where $x^2$ is a variable and not $x$ raised to a power. Similarly, the substitution $z = x^3$ is the replacement of $z$ by the variable $x^3$ and this should not be confused with $x$ raised to a power. In order to write a superscript quantity to a power, use parentheses. For example, $(x^2)^3$ is the variable $x^2$ cubed. One of the reasons for introducing the superscript variables is that many equations of mathematics and physics can be made to take on a concise and compact form.

There is a range convention associated with the indices. This convention states that whenever there is an expression where the indices occur unrepeated it is to be understood that each of the subscripts or superscripts can take on any of the integer values $1, 2, \dots, N$ where $N$ is a specified integer. For example, the Kronecker delta symbol $\delta_{ij}$, defined by $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ for $i \neq j$, with $i, j$ ranging over the values 1,2,3, represents the 9 quantities

$$\begin{array}{lll} \delta_{11} = 1 & \delta_{12} = 0 & \delta_{13} = 0 \\ \delta_{21} = 0 & \delta_{22} = 1 & \delta_{23} = 0 \\ \delta_{31} = 0 & \delta_{32} = 0 & \delta_{33} = 1. \end{array}$$

The symbol $\delta_{ij}$ refers to all of the components of the system simultaneously. As another example, consider the equation

$$\widehat{\mathbf{e}}_m \cdot \widehat{\mathbf{e}}_n = \delta_{mn} \qquad m, n = 1, 2, 3 \tag{1.1.1}$$

the subscripts $m, n$ occur unrepeated on the left side of the equation and hence must also occur on the right hand side of the equation. These indices are called "free "indices and can take on any of the values $1, 2$ or $3$ as specified by the range. Since there are three choices for the value for $m$ and three choices for a value of $n$ we find that equation (1.1.1) represents nine equations simultaneously. These nine equations are

$$\begin{array}{lll} \widehat{\mathbf{e}}_1 \cdot \widehat{\mathbf{e}}_1 = 1 & \widehat{\mathbf{e}}_1 \cdot \widehat{\mathbf{e}}_2 = 0 & \widehat{\mathbf{e}}_1 \cdot \widehat{\mathbf{e}}_3 = 0 \\ \widehat{\mathbf{e}}_2 \cdot \widehat{\mathbf{e}}_1 = 0 & \widehat{\mathbf{e}}_2 \cdot \widehat{\mathbf{e}}_2 = 1 & \widehat{\mathbf{e}}_2 \cdot \widehat{\mathbf{e}}_3 = 0 \\ \widehat{\mathbf{e}}_3 \cdot \widehat{\mathbf{e}}_1 = 0 & \widehat{\mathbf{e}}_3 \cdot \widehat{\mathbf{e}}_2 = 0 & \widehat{\mathbf{e}}_3 \cdot \widehat{\mathbf{e}}_3 = 1. \end{array}$$

**Symmetric and Skew-Symmetric Systems**

A system defined by subscripts and superscripts ranging over a set of values is said to be symmetric in two of its indices if the components are unchanged when the indices are interchanged. For example, the third order system $T_{ijk}$ is symmetric in the indices $i$ and $k$ if

$$T_{ijk} = T_{kji} \quad \text{for all values of } i, j \text{ and } k.$$

A system defined by subscripts and superscripts is said to be skew-symmetric in two of its indices if the components change sign when the indices are interchanged. For example, the fourth order system $T_{ijkl}$ is skew-symmetric in the indices $i$ and $l$ if

$$T_{ijkl} = -T_{ljki} \quad \text{for all values of } ijk \text{ and } l.$$

As another example, consider the third order system $a_{prs}$, $p, r, s = 1, 2, 3$ which is completely skew-symmetric in all of its indices. We would then have

$$a_{prs} = -a_{psr} = a_{spr} = -a_{srp} = a_{rsp} = -a_{rps}.$$

It is left as an exercise to show this completely skew- symmetric systems has 27 elements, 21 of which are zero. The 6 nonzero elements are all related to one another thru the above equations when $(p, r, s) = (1, 2, 3)$. This is expressed as saying that the above system has only one independent component.

**Summation Convention**

The summation convention states that whenever there arises an expression where there is an index which occurs twice on the same side of any equation, or term within an equation, it is understood to represent a summation on these repeated indices. The summation being over the integer values specified by the range. A repeated index is called a summation index, while an unrepeated index is called a free index. The summation convention requires that one must never allow a summation index to appear more than twice in any given expression. Because of this rule it is sometimes necessary to replace one dummy summation symbol by some other dummy symbol in order to avoid having three or more indices occurring on the same side of the equation. The index notation is a very powerful notation and can be used to concisely represent many complex equations. For the remainder of this section there is presented additional definitions and examples to illustrated the power of the indicial notation. This notation is then employed to define tensor components and associated operations with tensors.

**EXAMPLE 1.1-1** The two equations

$$y_1 = a_{11}x_1 + a_{12}x_2$$
$$y_2 = a_{21}x_1 + a_{22}x_2$$

can be represented as one equation by introducing a dummy index, say $k$, and expressing the above equations as

$$y_k = a_{k1}x_1 + a_{k2}x_2, \qquad k = 1, 2.$$

The range convention states that $k$ is free to have any one of the values 1 or 2, ($k$ is a free index). This equation can now be written in the form

$$y_k = \sum_{i=1}^{2} a_{ki}x_i = a_{k1}x_1 + a_{k2}x_2$$

where $i$ is the dummy summation index. When the summation sign is removed and the summation convention is adopted we have

$$y_k = a_{ki}x_i \qquad i, k = 1, 2.$$

Since the subscript $i$ repeats itself, the summation convention requires that a summation be performed by letting the summation subscript take on the values specified by the range and then summing the results. The index $k$ which appears only once on the left and only once on the right hand side of the equation is called a free index. It should be noted that both $k$ and $i$ are dummy subscripts and can be replaced by other letters. For example, we can write

$$y_n = a_{nm}x_m \qquad n, m = 1, 2$$

where $m$ is the summation index and $n$ is the free index. Summing on $m$ produces

$$y_n = a_{n1}x_1 + a_{n2}x_2$$

and letting the free index $n$ take on the values of 1 and 2 we produce the original two equations.

∎

**EXAMPLE 1.1-2.** For $y_i = a_{ij}x_j$, $i,j = 1,2,3$ and $x_i = b_{ij}z_j$, $i,j = 1,2,3$ solve for the $y$ variables in terms of the $z$ variables.

**Solution:** In matrix form the given equations can be expressed:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}.$$

Now solve for the $y$ variables in terms of the $z$ variables and obtain

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}.$$

The index notation employs indices that are dummy indices and so we can write

$$y_n = a_{nm}x_m, \quad n,m = 1,2,3 \quad \text{and} \quad x_m = b_{mj}z_j, \quad m,j = 1,2,3.$$

Here we have purposely changed the indices so that when we substitute for $x_m$, from one equation into the other, a summation index does not repeat itself more than twice. Substituting we find the indicial form of the above matrix equation as

$$y_n = a_{nm}b_{mj}z_j, \quad m,n,j = 1,2,3$$

where $n$ is the free index and $m,j$ are the dummy summation indices. It is left as an exercise to expand both the matrix equation and the indicial equation and verify that they are different ways of representing the same thing.

■

**EXAMPLE 1.1-3.** The dot product of two vectors $A_q$, $q = 1,2,3$ and $B_j$, $j = 1,2,3$ can be represented with the index notation by the product $A_iB_i = AB\cos\theta \quad i = 1,2,3, \quad A = |\vec{A}|, \quad B = |\vec{B}|$. Since the subscript $i$ is repeated it is understood to represent a summation index. Summing on $i$ over the range specified, there results

$$A_1B_1 + A_2B_2 + A_3B_3 = AB\cos\theta.$$

Observe that the index notation employs dummy indices. At times these indices are altered in order to conform to the above summation rules, without attention being brought to the change. As in this example, the indices $q$ and $j$ are dummy indices and can be changed to other letters if one desires. Also, in the future, if the range of the indices is not stated it is assumed that the range is over the integer values $1,2$ and $3$.

■

To systems containing subscripts and superscripts one can apply certain algebraic operations. We present in an informal way the operations of addition, multiplication and contraction.

**Addition, Multiplication and Contraction**

The algebraic operation of addition or subtraction applies to systems of the same type and order. That is we can add or subtract like components in systems. For example, the sum of $A^i_{jk}$ and $B^i_{jk}$ is again a system of the same type and is denoted by $C^i_{jk} = A^i_{jk} + B^i_{jk}$, where like components are added.

The product of two systems is obtained by multiplying each component of the first system with each component of the second system. Such a product is called an outer product. The order of the resulting product system is the sum of the orders of the two systems involved in forming the product. For example, if $A^i_j$ is a second order system and $B^{mnl}$ is a third order system, with all indices having the range 1 to N, then the product system is fifth order and is denoted $C^{imnl}_j = A^i_j B^{mnl}$. The product system represents $N^5$ terms constructed from all possible products of the components from $A^i_j$ with the components from $B^{mnl}$.

The operation of contraction occurs when a lower index is set equal to an upper index and the summation convention is invoked. For example, if we have a fifth order system $C^{imnl}_j$ and we set $i = j$ and sum, then we form the system

$$C^{mnl} = C^{jmnl}_j = C^{1mnl}_1 + C^{2mnl}_2 + \cdots + C^{Nmnl}_N.$$

Here the symbol $C^{mnl}$ is used to represent the third order system that results when the contraction is performed. Whenever a contraction is performed, the resulting system is always of order 2 less than the original system. Under certain special conditions it is permissible to perform a contraction on two lower case indices. These special conditions will be considered later in the section.

The above operations will be more formally defined after we have explained what tensors are.

**The e-permutation symbol and Kronecker delta**

Two symbols that are used quite frequently with the indicial notation are the e-permutation symbol and the Kronecker delta. The e-permutation symbol is sometimes referred to as the alternating tensor. The e-permutation symbol, as the name suggests, deals with permutations. A permutation is an arrangement of things. When the order of the arrangement is changed, a new permutation results. A transposition is an interchange of two consecutive terms in an arrangement. As an example, let us change the digits 1 2 3 to 3 2 1 by making a sequence of transpositions. Starting with the digits in the order 1 2 3 we interchange 2 and 3 (first transposition) to obtain 1 3 2. Next, interchange the digits 1 and 3 ( second transposition) to obtain 3 1 2. Finally, interchange the digits 1 and 2 (third transposition) to achieve 3 2 1. Here the total number of transpositions of 1 2 3 to 3 2 1 is three, an odd number. Other transpositions of 1 2 3 to 3 2 1 can also be written. However, these are also an odd number of transpositions.

**EXAMPLE 1.1-4.** The total number of possible ways of arranging the digits 1 2 3 is six. We have three choices for the first digit. Having chosen the first digit, there are only two choices left for the second digit. Hence the remaining number is for the last digit. The product $(3)(2)(1) = 3! = 6$ is the number of permutations of the digits $1, 2$ and 3. These six permutations are

| | |
|---|---|
| 1 2 3 | even permutation |
| 1 3 2 | odd permutation |
| 3 1 2 | even permutation |
| 3 2 1 | odd permutation |
| 2 3 1 | even permutation |
| 2 1 3 | odd permutation. |

Here a permutation of 1 2 3 is called even or odd depending upon whether there is an even or odd number of transpositions of the digits. A mnemonic device to remember the even and odd permutations of 123 is illustrated in the figure 1.1-1. Note that even permutations of 123 are obtained by selecting any three consecutive numbers from the sequence 123123 and the odd permutations result by selecting any three consecutive numbers from the sequence 321321.

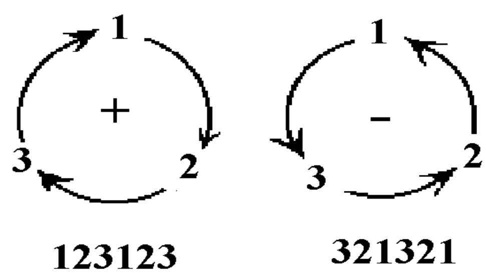


Figure 1.1-1. Permutations of 123.

■

In general, the number of permutations of $n$ things taken $m$ at a time is given by the relation

$$P(n,m) = n(n-1)(n-2)\cdots(n-m+1).$$

By selecting a subset of $m$ objects from a collection of $n$ objects, $m \le n$, without regard to the ordering is called a combination of $n$ objects taken $m$ at a time. For example, combinations of 3 numbers taken from the set $\{1, 2, 3, 4\}$ are $(123), (124), (134), (234)$. Note that ordering of a combination is not considered. That is, the permutations $(123), (132), (231), (213), (312), (321)$ are considered equal. In general, the number of combinations of $n$ objects taken $m$ at a time is given by $C(n,m) = \binom{n}{m} = \dfrac{n!}{m!(n-m)!}$ where $\binom{n}{m}$ are the binomial coefficients which occur in the expansion

$$(a+b)^n = \sum_{m=0}^{n} \binom{n}{m} a^{n-m} b^m.$$

The definition of permutations can be used to define the e-permutation symbol.

**Definition: (e-Permutation symbol or alternating tensor)**
The e-permutation symbol is defined

$$e^{ijk\dots l} = e_{ijk\dots l} = \begin{cases} 1 & \text{if } ijk\dots l \text{ is an even permutation of the integers } 123\dots n \\ -1 & \text{if } ijk\dots l \text{ is an odd permutation of the integers } 123\dots n \\ 0 & \text{in all other cases} \end{cases}$$

**EXAMPLE 1.1-5.** Find $e_{612453}$.
**Solution:** To determine whether 612453 is an even or odd permutation of 123456 we write down the given numbers and below them we write the integers 1 through 6. Like numbers are then connected by a line and we obtain figure 1.1-2.

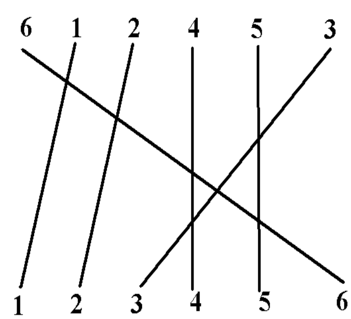


Figure 1.1-2. Permutations of 123456.

In figure 1.1-2, there are seven intersections of the lines connecting like numbers. The number of intersections is an odd number and shows that an odd number of transpositions must be performed. These results imply $e_{612453} = -1$.

■

Another definition used quite frequently in the representation of mathematical and engineering quantities is the Kronecker delta which we now define in terms of both subscripts and superscripts.

**Definition: (Kronecker delta)** The Kronecker delta is defined:

$$\delta_{ij} = \delta_i^j = \begin{cases} 1 & \text{if } i \text{ equals } j \\ 0 & \text{if } i \text{ is different from } j \end{cases}$$

**EXAMPLE 1.1-6.** Some examples of the $e-$permutation symbol and Kronecker delta are:

$$\begin{array}{lll} e_{123} = e^{123} = +1 & \delta^1_1 = 1 & \delta_{12} = 0 \\ e_{213} = e^{213} = -1 & \delta^1_2 = 0 & \delta_{22} = 1 \\ e_{112} = e^{112} = 0 & \delta^1_3 = 0 & \delta_{32} = 0. \end{array}$$

■

**EXAMPLE 1.1-7.** When an index of the Kronecker delta $\delta_{ij}$ is involved in the summation convention, the effect is that of replacing one index with a different index. For example, let $a_{ij}$ denote the elements of an $N \times N$ matrix. Here $i$ and $j$ are allowed to range over the integer values $1, 2, \ldots, N$. Consider the product

$$a_{ij}\delta_{ik}$$

where the range of $i,\ j,\ k$ is $1, 2, \ldots, N$. The index $i$ is repeated and therefore it is understood to represent a summation over the range. The index $i$ is called a summation index. The other indices $j$ and $k$ are free indices. They are free to be assigned any values from the range of the indices. They are not involved in any summations and their values, whatever you choose to assign them, are fixed. Let us assign a value of $\underline{j}$ and $\underline{k}$ to the values of $j$ and $k$. The underscore is to remind you that these values for $j$ and $k$ are fixed and not to be summed. When we perform the summation over the summation index $i$ we assign values to $i$ from the range and then sum over these values. Performing the indicated summation we obtain

$$a_{i\underline{j}}\delta_{i\underline{k}} = a_{1\underline{j}}\delta_{1\underline{k}} + a_{2\underline{j}}\delta_{2\underline{k}} + \cdots + a_{\underline{k}\underline{j}}\delta_{\underline{k}\underline{k}} + \cdots + a_{N\underline{j}}\delta_{N\underline{k}}.$$

In this summation the Kronecker delta is zero everywhere the subscripts are different and equals one where the subscripts are the same. There is only one term in this summation which is nonzero. It is that term where the summation index $i$ was equal to the fixed value $\underline{k}$ This gives the result

$$a_{\underline{k}\underline{j}}\delta_{\underline{k}\underline{k}} = a_{\underline{k}\underline{j}}$$

where the underscore is to remind you that the quantities have fixed values and are not to be summed. Dropping the underscores we write

$$a_{ij}\delta_{ik} = a_{kj}$$

Here we have substituted the index $i$ by $k$ and so when the Kronecker delta is used in a summation process it is known as a substitution operator. This substitution property of the Kronecker delta can be used to simplify a variety of expressions involving the index notation. Some examples are:

$$\begin{gathered} B_{ij}\delta_{js} = B_{is} \\ \delta_{jk}\delta_{km} = \delta_{jm} \\ e_{ijk}\delta_{im}\delta_{jn}\delta_{kp} = e_{mnp}. \end{gathered}$$

Some texts adopt the notation that if indices are capital letters, then no summation is to be performed. For example,

$$a_{KJ}\delta_{KK} = a_{KJ}$$

as $\delta_{KK}$ represents a single term because of the capital letters. Another notation which is used to denote no summation of the indices is to put parenthesis about the indices which are not to be summed. For example,

$$a_{(k)j}\delta_{(k)(k)} = a_{kj},$$

since $\delta_{(k)(k)}$ represents a single term and the parentheses indicate that no summation is to be performed. At any time we may employ either the underscore notation, the capital letter notation or the parenthesis notation to denote that no summation of the indices is to be performed. To avoid confusion altogether, one can write out parenthetical expressions such as "(no summation on $k$)".

■

**EXAMPLE 1.1-8.** In the Kronecker delta symbol $\delta^i_j$ we set $j$ equal to $i$ and perform a summation. This operation is called a contraction. There results $\delta^i_i$, which is to be summed over the range of the index $i$. Utilizing the range $1, 2, \ldots, N$ we have

$$\begin{aligned}
\delta^i_i &= \delta^1_1 + \delta^2_2 + \cdots + \delta^N_N \\
\delta^i_i &= 1 + 1 + \cdots + 1 \\
\delta^i_i &= N.
\end{aligned}$$

In three dimension we have $\delta^i_j$, $i, j = 1, 2, 3$ and

$$\delta^k_k = \delta^1_1 + \delta^2_2 + \delta^3_3 = 3.$$

In certain circumstances the Kronecker delta can be written with only subscripts. For example, $\delta_{ij}, \quad i, j = 1, 2, 3$. We shall find that these circumstances allow us to perform a contraction on the lower indices so that $\delta_{ii} = 3$.

■

**EXAMPLE 1.1-9.** The determinant of a matrix $A = (a_{ij})$ can be represented in the indicial notation. Employing the e-permutation symbol the determinant of an $N \times N$ matrix is expressed

$$|A| = e_{ij\ldots k} a_{1i} a_{2j} \cdots a_{Nk}$$

where $e_{ij\ldots k}$ is an $N$th order system. In the special case of a $2 \times 2$ matrix we write

$$|A| = e_{ij} a_{1i} a_{2j}$$

where the summation is over the range 1,2 and the e-permutation symbol is of order 2. In the special case of a $3 \times 3$ matrix we have

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = e_{ijk} a_{i1} a_{j2} a_{k3} = e_{ijk} a_{1i} a_{2j} a_{3k}$$

where $i, j, k$ are the summation indices and the summation is over the range 1,2,3. Here $e_{ijk}$ denotes the e-permutation symbol of order 3. Note that by interchanging the rows of the $3 \times 3$ matrix we can obtain more general results. Consider $(p, q, r)$ as some permutation of the integers $(1, 2, 3)$, and observe that the determinant can be expressed

$$\Delta = \begin{vmatrix} a_{p1} & a_{p2} & a_{p3} \\ a_{q1} & a_{q2} & a_{q3} \\ a_{r1} & a_{r2} & a_{r3} \end{vmatrix} = e_{ijk} a_{pi} a_{qj} a_{rk}.$$

If $(p, q, r)$ is an even permutation of $(1, 2, 3)$ then $\Delta = |A|$

If $(p, q, r)$ is an odd permutation of $(1, 2, 3)$ then $\Delta = -|A|$

If $(p, q, r)$ is not a permutation of $(1, 2, 3)$ then $\Delta = 0.$

We can then write

$$e_{ijk} a_{pi} a_{qj} a_{rk} = e_{pqr} |A|.$$

Each of the above results can be verified by performing the indicated summations. A more formal proof of the above result is given in EXAMPLE 1.1-25, later in this section.

■

**EXAMPLE 1.1-10.** The expression $e_{ijk} B_{ij} C_i$ is meaningless since the index $i$ repeats itself more than twice and the summation convention does not allow this. If you really did want to sum over an index which occurs more than twice, then one must use a summation sign. For example the above expression would be written $\sum_{i=1}^{n} e_{ijk} B_{ij} C_i$.

■

**EXAMPLE 1.1-11.**

The cross product of the unit vectors $\widehat{\mathbf{e}}_1$, $\widehat{\mathbf{e}}_2$, $\widehat{\mathbf{e}}_3$ can be represented in the index notation by

$$\widehat{\mathbf{e}}_i \times \widehat{\mathbf{e}}_j = \begin{cases} \widehat{\mathbf{e}}_k & \text{if } (i, j, k) \text{ is an even permutation of } (1, 2, 3) \\ -\widehat{\mathbf{e}}_k & \text{if } (i, j, k) \text{ is an odd permutation of } (1, 2, 3) \\ 0 & \text{in all other cases} \end{cases}$$

This result can be written in the form $\widehat{\mathbf{e}}_i \times \widehat{\mathbf{e}}_j = e_{kij} \widehat{\mathbf{e}}_k$. This later result can be verified by summing on the index $k$ and writing out all 9 possible combinations for $i$ and $j$.

■

**EXAMPLE 1.1-12.** Given the vectors $A_p$, $p = 1, 2, 3$ and $B_p$, $p = 1, 2, 3$ the cross product of these two vectors is a vector $C_p$, $p = 1, 2, 3$ with components

$$C_i = e_{ijk} A_j B_k, \quad i, j, k = 1, 2, 3. \tag{1.1.2}$$

The quantities $C_i$ represent the components of the cross product vector

$$\vec{C} = \vec{A} \times \vec{B} = C_1 \widehat{\mathbf{e}}_1 + C_2 \widehat{\mathbf{e}}_2 + C_3 \widehat{\mathbf{e}}_3.$$

The equation (1.1.2), which defines the components of $\vec{C}$, is to be summed over each of the indices which repeats itself. We have summing on the index $k$

$$C_i = e_{ij1} A_j B_1 + e_{ij2} A_j B_2 + e_{ij3} A_j B_3. \tag{1.1.3}$$

We next sum on the index $j$ which repeats itself in each term of equation (1.1.3). This gives

$$\begin{aligned} C_i = & e_{i11}A_1B_1 + e_{i21}A_2B_1 + e_{i31}A_3B_1 \\ & + e_{i12}A_1B_2 + e_{i22}A_2B_2 + e_{i32}A_3B_2 \\ & + e_{i13}A_1B_3 + e_{i23}A_2B_3 + e_{i33}A_3B_3. \end{aligned} \tag{1.1.4}$$

Now we are left with $i$ being a free index which can have any of the values of $1, 2$ or 3. Letting $i = 1$, then letting $i = 2$, and finally letting $i = 3$ produces the cross product components

$$\begin{aligned} C_1 &= A_2B_3 - A_3B_2 \\ C_2 &= A_3B_1 - A_1B_3 \\ C_3 &= A_1B_2 - A_2B_1. \end{aligned}$$

The cross product can also be expressed in the form $\vec{A} \times \vec{B} = e_{ijk}A_jB_k\,\widehat{\mathbf{e}}_i$. This result can be verified by summing over the indices $i$,$j$ and $k$.

■

**EXAMPLE 1.1-13.** Show

$$e_{ijk} = -e_{ikj} = e_{jki} \quad \text{for} \quad i, j, k = 1, 2, 3$$

**Solution:** The array $i\ k\ j$ represents an odd number of transpositions of the indices $i\ j\ k$ and to each transposition there is a sign change of the e-permutation symbol. Similarly, $j\ k\ i$ is an even transposition of $i\ j\ k$ and so there is no sign change of the e-permutation symbol. The above holds regardless of the numerical values assigned to the indices $i, j, k$.

■

**The e-$\delta$ Identity**

An identity relating the e-permutation symbol and the Kronecker delta, which is useful in the simplification of tensor expressions, is the e-$\delta$ identity. This identity can be expressed in different forms. The subscript form for this identity is

$$e_{ijk}e_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}, \qquad i, j, k, m, n = 1, 2, 3$$

where $i$ is the summation index and $j, k, m, n$ are free indices. A device used to remember the positions of the subscripts is given in the figure 1.1-3.

The subscripts on the four Kronecker delta's on the right-hand side of the e-$\delta$ identity then are read

(first)(second)-(outer)(inner).

This refers to the positions following the summation index. Thus, $j, m$ are the first indices after the summation index and $k, n$ are the second indices after the summation index. The indices $j, n$ are outer indices when compared to the inner indices $k, m$ as the indices are viewed as written on the left-hand side of the identity.

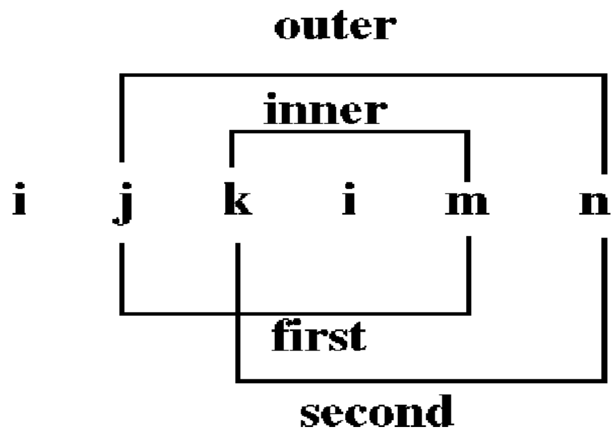


Figure 1.1-3. Mnemonic device for position of subscripts.

Another form of this identity employs both subscripts and superscripts and has the form

$$e^{ijk}e_{imn} = \delta^j_m\delta^k_n - \delta^j_n\delta^k_m. \tag{1.1.5}$$

One way of proving this identity is to observe the equation (1.1.5) has the free indices $j,k,m,n$. Each of these indices can have any of the values of $1,2$ or 3. There are 3 choices we can assign to each of $j,k,m$ or $n$ and this gives a total of $3^4 = 81$ possible equations represented by the identity from equation (1.1.5). By writing out all 81 of these equations we can verify that the identity is true for all possible combinations that can be assigned to the free indices.

An alternate proof of the $e-\delta$ identity is to consider the determinant

$$\begin{vmatrix} \delta^1_1 & \delta^1_2 & \delta^1_3 \\ \delta^2_1 & \delta^2_2 & \delta^2_3 \\ \delta^3_1 & \delta^3_2 & \delta^3_3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

By performing a permutation of the rows of this matrix we can use the permutation symbol and write

$$\begin{vmatrix} \delta^i_1 & \delta^i_2 & \delta^i_3 \\ \delta^j_1 & \delta^j_2 & \delta^j_3 \\ \delta^k_1 & \delta^k_2 & \delta^k_3 \end{vmatrix} = e^{ijk}.$$

By performing a permutation of the columns, we can write

$$\begin{vmatrix} \delta^i_r & \delta^i_s & \delta^i_t \\ \delta^j_r & \delta^j_s & \delta^j_t \\ \delta^k_r & \delta^k_s & \delta^k_t \end{vmatrix} = e^{ijk}e_{rst}.$$

Now perform a contraction on the indices $i$ and $r$ to obtain

$$\begin{vmatrix} \delta^i_i & \delta^i_s & \delta^i_t \\ \delta^j_i & \delta^j_s & \delta^j_t \\ \delta^k_i & \delta^k_s & \delta^k_t \end{vmatrix} = e^{ijk}e_{ist}.$$

Summing on $i$ we have $\delta^i_i = \delta^1_1 + \delta^2_2 + \delta^3_3 = 3$ and expand the determinant to obtain the desired result

$$\delta^j_s\delta^k_t - \delta^j_t\delta^k_s = e^{ijk}e_{ist}.$$

**Generalized Kronecker delta**

The generalized Kronecker delta is defined by the $(n \times n)$ determinant

$$\delta^{ij\ldots k}_{mn\ldots p} = \begin{vmatrix} \delta^i_m & \delta^i_n & \cdots & \delta^i_p \\ \delta^j_m & \delta^j_n & \cdots & \delta^j_p \\ \vdots & \vdots & \ddots & \vdots \\ \delta^k_m & \delta^k_n & \cdots & \delta^k_p \end{vmatrix}.$$

For example, in three dimensions we can write

$$\delta^{ijk}_{mnp} = \begin{vmatrix} \delta^i_m & \delta^i_n & \delta^i_p \\ \delta^j_m & \delta^j_n & \delta^j_p \\ \delta^k_m & \delta^k_n & \delta^k_p \end{vmatrix} = e^{ijk} e_{mnp}.$$

Performing a contraction on the indices $k$ and $p$ we obtain the fourth order system

$$\delta^{rs}_{mn} = \delta^{rsp}_{mnp} = e^{rsp} e_{mnp} = e^{prs} e_{pmn} = \delta^r_m \delta^s_n - \delta^r_n \delta^s_m.$$

As an exercise one can verify that the definition of the e-permutation symbol can also be defined in terms of the generalized Kronecker delta as

$$e_{j_1 j_2 j_3 \cdots j_N} = \delta^{1\ 2\ 3 \cdots N}_{j_1 j_2 j_3 \cdots j_N}.$$

Additional definitions and results employing the generalized Kronecker delta are found in the exercises. In section 1.3 we shall show that the Kronecker delta and epsilon permutation symbol are numerical tensors which have fixed components in every coordinate system.

**Additional Applications of the Indicial Notation**

The indicial notation, together with the $e - \delta$ identity, can be used to prove various vector identities.

**EXAMPLE 1.1-14.** Show, using the index notation, that $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

**Solution:** Let

$$\vec{C} = \vec{A} \times \vec{B} = C_1\,\widehat{\mathbf{e}}_1 + C_2\,\widehat{\mathbf{e}}_2 + C_3\,\widehat{\mathbf{e}}_3 = C_i\,\widehat{\mathbf{e}}_i \qquad \text{and let}$$
$$\vec{D} = \vec{B} \times \vec{A} = D_1\,\widehat{\mathbf{e}}_1 + D_2\,\widehat{\mathbf{e}}_2 + D_3\,\widehat{\mathbf{e}}_3 = D_i\,\widehat{\mathbf{e}}_i.$$

We have shown that the components of the cross products can be represented in the index notation by

$$C_i = e_{ijk} A_j B_k \quad \text{and} \quad D_i = e_{ijk} B_j A_k.$$

We desire to show that $D_i = -C_i$ for all values of $i$. Consider the following manipulations: Let $B_j = B_s \delta_{sj}$ and $A_k = A_m \delta_{mk}$ and write

$$D_i = e_{ijk} B_j A_k = e_{ijk} B_s \delta_{sj} A_m \delta_{mk} \tag{1.1.6}$$

where all indices have the range $1, 2, 3$. In the expression (1.1.6) note that no summation index appears more than twice because if an index appeared more than twice the summation convention would become meaningless. By rearranging terms in equation (1.1.6) we have

$$D_i = e_{ijk} \delta_{sj} \delta_{mk} B_s A_m = e_{ism} B_s A_m.$$

In this expression the indices $s$ and $m$ are dummy summation indices and can be replaced by any other letters. We replace $s$ by $k$ and $m$ by $j$ to obtain

$$D_i = e_{ikj}A_jB_k = -e_{ijk}A_jB_k = -C_i.$$

Consequently, we find that $\vec{D} = -\vec{C}$ or $\vec{B} \times \vec{A} = -\vec{A} \times \vec{B}$. That is, $\vec{D} = D_i\,\widehat{\mathbf{e}}_i = -C_i\,\widehat{\mathbf{e}}_i = -\vec{C}$.

Note 1. The expressions

$$C_i = e_{ijk}A_jB_k \qquad \text{and} \qquad C_m = e_{mnp}A_nB_p$$

with all indices having the range $1, 2, 3$, appear to be different because different letters are used as subscripts. It must be remembered that certain indices are summed according to the summation convention and the other indices are free indices and can take on any values from the assigned range. Thus, after summation, when numerical values are substituted for the indices involved, none of the dummy letters used to represent the components appear in the answer.

Note 2. A second important point is that when one is working with expressions involving the index notation, the indices can be changed directly. For example, in the above expression for $D_i$ we could have replaced $j$ by $k$ and $k$ by $j$ simultaneously (so that no index repeats itself more than twice) to obtain

$$D_i = e_{ijk}B_jA_k = e_{ikj}B_kA_j = -e_{ijk}A_jB_k = -C_i.$$

Note 3. Be careful in switching back and forth between the vector notation and index notation. Observe that a vector $\vec{A}$ can be represented

$$\vec{A} = A_i\,\widehat{\mathbf{e}}_i$$

or its components can be represented

$$\vec{A} \cdot\, \widehat{\mathbf{e}}_i = A_i, \quad i = 1, 2, 3.$$

Do not set a vector equal to a scalar. That is, do not make the mistake of writing $\vec{A} = A_i$ as this is a misuse of the equal sign. It is not possible for a vector to equal a scalar because they are two entirely different quantities. A vector has both magnitude and direction while a scalar has only magnitude.

■

**EXAMPLE 1.1-15.** Verify the vector identity

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A})$$

**Solution:** Let

$$\vec{B} \times \vec{C} = \vec{D} = D_i\,\widehat{\mathbf{e}}_i \qquad \text{where} \qquad D_i = e_{ijk}B_jC_k \qquad \text{and let}$$
$$\vec{C} \times \vec{A} = \vec{F} = F_i\,\widehat{\mathbf{e}}_i \qquad \text{where} \qquad F_i = e_{ijk}C_jA_k$$

where all indices have the range $1, 2, 3$. To prove the above identity, we have

$$\begin{aligned} \vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{A} \cdot \vec{D} = A_iD_i &= A_ie_{ijk}B_jC_k \\ &= B_j(e_{ijk}A_iC_k) \\ &= B_j(e_{jki}C_kA_i) \end{aligned}$$

since $e_{ijk} = e_{jki}$. We also observe from the expression

$$F_i = e_{ijk} C_j A_k$$

that we may obtain, by permuting the symbols, the equivalent expression

$$F_j = e_{jki} C_k A_i.$$

This allows us to write

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = B_j F_j = \vec{B} \cdot \vec{F} = \vec{B} \cdot (\vec{C} \times \vec{A})$$

which was to be shown.

The quantity $\vec{A} \cdot (\vec{B} \times \vec{C})$ is called a triple scalar product. The above index representation of the triple scalar product implies that it can be represented as a determinant (See example 1.1-9). We can write

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = e_{ijk} A_i B_j C_k$$

A physical interpretation that can be assigned to this triple scalar product is that its absolute value represents the volume of the parallelepiped formed by the three noncoplaner vectors $\vec{A}, \vec{B}, \vec{C}$. The absolute value is needed because sometimes the triple scalar product is negative. This physical interpretation can be obtained from an analysis of the figure 1.1-4.

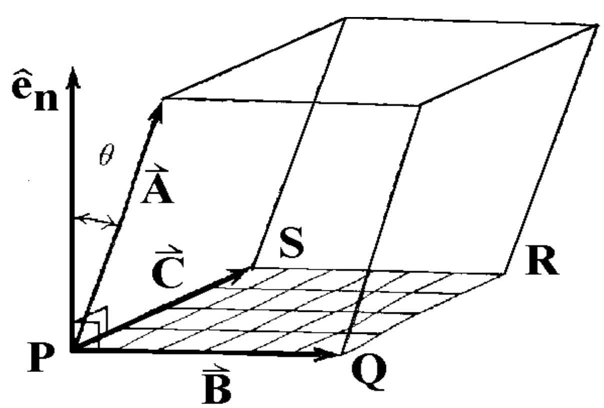


Figure 1.1-4. Triple scalar product and volume

In figure 1.1-4 observe that: (i) $|\vec{B}\times\vec{C}|$ is the area of the parallelogram $PQRS$. (ii) the unit vector

$$\widehat{\mathbf{e}}_n = \frac{\vec{B}\times\vec{C}}{|\vec{B}\times\vec{C}|}$$

is normal to the plane containing the vectors $\vec{B}$ and $\vec{C}$. (iii) The dot product

$$\left|\vec{A}\cdot\widehat{\mathbf{e}}_n\right| = \left|\vec{A}\cdot\frac{\vec{B}\times\vec{C}}{|\vec{B}\times\vec{C}|}\right| = h$$

equals the projection of $\vec{A}$ on $\widehat{\mathbf{e}}_n$ which represents the height of the parallelepiped. These results demonstrate that

$$\left|\vec{A}\cdot(\vec{B}\times\vec{C})\right| = |\vec{B}\times\vec{C}|\,h = \text{(area of base)(height)} = \text{volume}.$$

■

**EXAMPLE 1.1-16.** Verify the vector identity

$$(\vec{A}\times\vec{B})\times(\vec{C}\times\vec{D}) = \vec{C}(\vec{D}\cdot\vec{A}\times\vec{B}) - \vec{D}(\vec{C}\cdot\vec{A}\times\vec{B})$$

**Solution:** Let $\vec{F} = \vec{A}\times\vec{B} = F_i\,\widehat{\mathbf{e}}_i$ and $\vec{E} = \vec{C}\times\vec{D} = E_i\,\widehat{\mathbf{e}}_i$. These vectors have the components

$$F_i = e_{ijk}A_jB_k \qquad \text{and} \qquad E_m = e_{mnp}C_nD_p$$

where all indices have the range $1,2,3$. The vector $\vec{G} = \vec{F}\times\vec{E} = G_i\,\widehat{\mathbf{e}}_i$ has the components

$$G_q = e_{qim}F_iE_m = e_{qim}e_{ijk}e_{mnp}A_jB_kC_nD_p.$$

From the identity $e_{qim} = e_{mqi}$ this can be expressed

$$G_q = (e_{mqi}e_{mnp})e_{ijk}A_jB_kC_nD_p$$

which is now in a form where we can use the $e-\delta$ identity applied to the term in parentheses to produce

$$G_q = (\delta_{qn}\delta_{ip} - \delta_{qp}\delta_{in})e_{ijk}A_jB_kC_nD_p.$$

Simplifying this expression we have:

$$\begin{aligned} G_q &= e_{ijk}\left[(D_p\delta_{ip})(C_n\delta_{qn})A_jB_k - (D_p\delta_{qp})(C_n\delta_{in})A_jB_k\right] \\ &= e_{ijk}\left[D_iC_qA_jB_k - D_qC_iA_jB_k\right] \\ &= C_q\left[D_ie_{ijk}A_jB_k\right] - D_q\left[C_ie_{ijk}A_jB_k\right] \end{aligned}$$

which are the vector components of the vector

$$\vec{C}(\vec{D}\cdot\vec{A}\times\vec{B}) - \vec{D}(\vec{C}\cdot\vec{A}\times\vec{B}).$$

■

**Transformation Equations**

Consider two sets of $N$ independent variables which are denoted by the barred and unbarred symbols $\overline{x}^i$ and $x^i$ with $i = 1, \ldots, N$. The independent variables $x^i, i = 1, \ldots, N$ can be thought of as defining the coordinates of a point in a $N-$dimensional space. Similarly, the independent barred variables define a point in some other $N-$dimensional space. These coordinates are assumed to be real quantities and are not complex quantities. Further, we assume that these variables are related by a set of transformation equations.

$$x^i = x^i(\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^N) \qquad i = 1, \ldots, N. \tag{1.1.7}$$

It is assumed that these transformation equations are independent. A necessary and sufficient condition that these transformation equations be independent is that the Jacobian determinant be different from zero, that is

$$J(\frac{x}{\overline{x}}) = \left|\frac{\partial x^i}{\partial \overline{x}^j}\right| = \begin{vmatrix} \frac{\partial x^1}{\partial \overline{x}^1} & \frac{\partial x^1}{\partial \overline{x}^2} & \cdots & \frac{\partial x^1}{\partial \overline{x}^N} \\ \frac{\partial x^2}{\partial \overline{x}^1} & \frac{\partial x^2}{\partial \overline{x}^2} & \cdots & \frac{\partial x^2}{\partial \overline{x}^N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x^N}{\partial \overline{x}^1} & \frac{\partial x^N}{\partial \overline{x}^2} & \cdots & \frac{\partial x^N}{\partial \overline{x}^N} \end{vmatrix} \neq 0.$$

This assumption allows us to obtain a set of inverse relations

$$\overline{x}^i = \overline{x}^i(x^1, x^2, \ldots, x^N) \qquad i = 1, \ldots, N, \tag{1.1.8}$$

where the $\overline{x}'s$ are determined in terms of the $x's$. Throughout our discussions it is to be understood that the given transformation equations are real and continuous. Further all derivatives that appear in our discussions are assumed to exist and be continuous in the domain of the variables considered.

**EXAMPLE 1.1-17.** The following is an example of a set of transformation equations of the form defined by equations (1.1.7) and (1.1.8) in the case $N = 3$. Consider the transformation from cylindrical coordinates $(r, \alpha, z)$ to spherical coordinates $(\rho, \beta, \alpha)$. From the geometry of the figure 1.1-5 we can find the transformation equations

$$\begin{aligned} r &= \rho \sin\beta \\ \alpha &= \alpha \qquad 0 < \alpha < 2\pi \\ z &= \rho\cos\beta \qquad 0 < \beta < \pi \end{aligned}$$

with inverse transformation

$$\begin{aligned} \rho &= \sqrt{r^2 + z^2} \\ \alpha &= \alpha \\ \beta &= \arctan(\frac{r}{z}) \end{aligned}$$

Now make the substitutions

$$(x^1, x^2, x^3) = (r, \alpha, z) \qquad \text{and} \qquad (\overline{x}^1, \overline{x}^2, \overline{x}^3) = (\rho, \beta, \alpha).$$

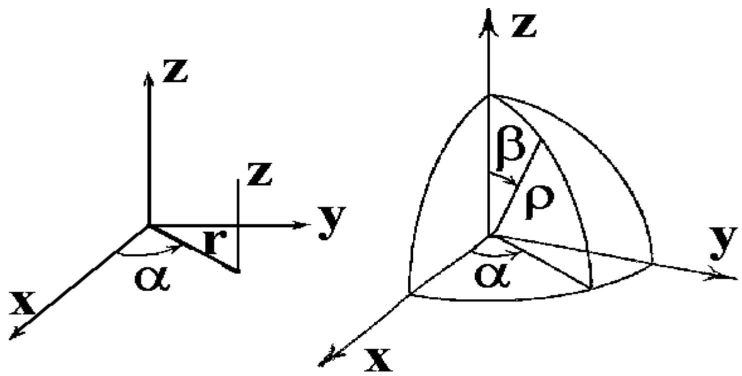


Figure 1.1-5. Cylindrical and Spherical Coordinates

The resulting transformations then have the forms of the equations (1.1.7) and (1.1.8).

■

**Calculation of Derivatives**

We now consider the chain rule applied to the differentiation of a function of the bar variables. We represent this differentiation in the indicial notation. Let $\Phi = \Phi(\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^n)$ be a scalar function of the variables $\overline{x}^i, \quad i = 1, \ldots, N$ and let these variables be related to the set of variables $x^i$, with $i = 1, \ldots, N$ by the transformation equations (1.1.7) and (1.1.8). The partial derivatives of $\Phi$ with respect to the variables $x^i$ can be expressed in the indicial notation as

$$\frac{\partial \Phi}{\partial x^i} = \frac{\partial \Phi}{\partial \overline{x}^j} \frac{\partial \overline{x}^j}{\partial x^i} = \frac{\partial \Phi}{\partial \overline{x}^1} \frac{\partial \overline{x}^1}{\partial x^i} + \frac{\partial \Phi}{\partial \overline{x}^2} \frac{\partial \overline{x}^2}{\partial x^i} + \cdots + \frac{\partial \Phi}{\partial \overline{x}^N} \frac{\partial \overline{x}^N}{\partial x^i} \tag{1.1.9}$$

for any fixed value of $i$ satisfying $1 \le i \le N$.

The second partial derivatives of $\Phi$ can also be expressed in the index notation. Differentiation of equation (1.1.9) partially with respect to $x^m$ produces

$$\frac{\partial^2 \Phi}{\partial x^i \partial x^m} = \frac{\partial \Phi}{\partial \overline{x}^j} \frac{\partial^2 \overline{x}^j}{\partial x^i \partial x^m} + \frac{\partial}{\partial x^m} \left[ \frac{\partial \Phi}{\partial \overline{x}^j} \right] \frac{\partial \overline{x}^j}{\partial x^i}. \tag{1.1.10}$$

This result is nothing more than an application of the general rule for differentiating a product of two quantities. To evaluate the derivative of the bracketed term in equation (1.1.10) it must be remembered that the quantity inside the brackets is a function of the bar variables. Let

$$G = \frac{\partial \Phi}{\partial \overline{x}^j} = G(\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^N)$$

to emphasize this dependence upon the bar variables, then the derivative of $G$ is

$$\frac{\partial G}{\partial x^m} = \frac{\partial G}{\partial \overline{x}^k} \frac{\partial \overline{x}^k}{\partial x^m} = \frac{\partial^2 \Phi}{\partial \overline{x}^j \partial \overline{x}^k} \frac{\partial \overline{x}^k}{\partial x^m}. \tag{1.1.11}$$

This is just an application of the basic rule from equation (1.1.9) with $\Phi$ replaced by $G$. Hence the derivative from equation (1.1.10) can be expressed

$$\frac{\partial^2 \Phi}{\partial x^i \partial x^m} = \frac{\partial \Phi}{\partial \overline{x}^j} \frac{\partial^2 \overline{x}^j}{\partial x^i \partial x^m} + \frac{\partial^2 \Phi}{\partial \overline{x}^j \partial \overline{x}^k} \frac{\partial \overline{x}^j}{\partial x^i} \frac{\partial \overline{x}^k}{\partial x^m} \tag{1.1.12}$$

where $i, m$ are free indices and $j, k$ are dummy summation indices.

**EXAMPLE 1.1-18.** Let $\Phi = \Phi(r,\theta)$ where $r,\theta$ are polar coordinates related to the Cartesian coordinates $(x,y)$ by the transformation equations $x = r\cos\theta \qquad y = r\sin\theta$. Find the partial derivatives $\dfrac{\partial \Phi}{\partial x}$ and $\dfrac{\partial^2 \Phi}{\partial x^2}$

**Solution:** The partial derivative of $\Phi$ with respect to $x$ is found from the relation (1.1.9) and can be written

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \Phi}{\partial \theta}\frac{\partial \theta}{\partial x}. \tag{1.1.13}$$

The second partial derivative is obtained by differentiating the first partial derivative. From the product rule for differentiation we can write

$$\frac{\partial^2 \Phi}{\partial x^2} = \frac{\partial \Phi}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial r}{\partial x}\frac{\partial}{\partial x}\left[\frac{\partial \Phi}{\partial r}\right] + \frac{\partial \Phi}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial x}\left[\frac{\partial \Phi}{\partial \theta}\right]. \tag{1.1.14}$$

To further simplify (1.1.14) it must be remembered that the terms inside the brackets are to be treated as functions of the variables $r$ and $\theta$ and that the derivative of these terms can be evaluated by reapplying the basic rule from equation (1.1.13) with $\Phi$ replaced by $\frac{\partial \Phi}{\partial r}$ and then $\Phi$ replaced by $\frac{\partial \Phi}{\partial \theta}$. This gives

$$\begin{aligned}\frac{\partial^2 \Phi}{\partial x^2} =& \frac{\partial \Phi}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial r}{\partial x}\left[\frac{\partial^2 \Phi}{\partial r^2}\frac{\partial r}{\partial x} + \frac{\partial^2 \Phi}{\partial r \partial \theta}\frac{\partial \theta}{\partial x}\right] \\ &+ \frac{\partial \Phi}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial \theta}{\partial x}\left[\frac{\partial^2 \Phi}{\partial \theta \partial r}\frac{\partial r}{\partial x} + \frac{\partial^2 \Phi}{\partial \theta^2}\frac{\partial \theta}{\partial x}\right].\end{aligned} \tag{1.1.15}$$

From the transformation equations we obtain the relations $r^2 = x^2 + y^2$ and $\tan\theta = \dfrac{y}{x}$ and from these relations we can calculate all the necessary derivatives needed for the simplification of the equations (1.1.13) and (1.1.15). These derivatives are:

$$\begin{aligned} 2r\frac{\partial r}{\partial x} &= 2x \quad \text{or} \quad \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta \\ \sec^2\theta \frac{\partial \theta}{\partial x} &= -\frac{y}{x^2} \quad \text{or} \quad \frac{\partial \theta}{\partial x} = -\frac{y}{r^2} = -\frac{\sin\theta}{r} \\ \frac{\partial^2 r}{\partial x^2} = -\sin\theta\frac{\partial \theta}{\partial x} &= \frac{\sin^2\theta}{r} \qquad \frac{\partial^2 \theta}{\partial x^2} = \frac{-r\cos\theta\frac{\partial \theta}{\partial x} + \sin\theta\frac{\partial r}{\partial x}}{r^2} = \frac{2\sin\theta\cos\theta}{r^2}.\end{aligned}$$

Therefore, the derivatives from equations (1.1.13) and (1.1.15) can be expressed in the form

$$\begin{aligned}\frac{\partial \Phi}{\partial x} &= \frac{\partial \Phi}{\partial r}\cos\theta - \frac{\partial \Phi}{\partial \theta}\frac{\sin\theta}{r} \\ \frac{\partial^2 \Phi}{\partial x^2} &= \frac{\partial \Phi}{\partial r}\frac{\sin^2\theta}{r} + 2\frac{\partial \Phi}{\partial \theta}\frac{\sin\theta\cos\theta}{r^2} + \frac{\partial^2 \Phi}{\partial r^2}\cos^2\theta - 2\frac{\partial^2 \Phi}{\partial r \partial \theta}\frac{\cos\theta\sin\theta}{r} + \frac{\partial^2 \Phi}{\partial \theta^2}\frac{\sin^2\theta}{r^2}.\end{aligned}$$

By letting $\overline{x}^1 = r,\ \overline{x}^2 = \theta,\ x^1 = x,\ x^2 = y$ and performing the indicated summations in the equations (1.1.9) and (1.1.12) there is produced the same results as above.

■

## Vector Identities in Cartesian Coordinates

Employing the substitutions $x^1 = x,\ x^2 = y,\ x^3 = z$, where superscript variables are employed and denoting the unit vectors in Cartesian coordinates by $\widehat{\mathbf{e}}_1,\ \widehat{\mathbf{e}}_2,\ \widehat{\mathbf{e}}_3$, we illustrated how various vector operations are written by using the index notation.

**Gradient.** In Cartesian coordinates the gradient of a scalar field is

$$\text{grad}\,\phi = \frac{\partial\phi}{\partial x}\,\widehat{\mathbf{e}}_1 + \frac{\partial\phi}{\partial y}\,\widehat{\mathbf{e}}_2 + \frac{\partial\phi}{\partial z}\,\widehat{\mathbf{e}}_3.$$

The index notation focuses attention only on the components of the gradient. In Cartesian coordinates these components are represented using a comma subscript to denote the derivative

$$\widehat{\mathbf{e}}_j \cdot \text{grad}\,\phi = \phi_{,j} = \frac{\partial\phi}{\partial x^j}, \quad j = 1, 2, 3.$$

The comma notation will be discussed in section 4. For now we use it to denote derivatives. For example $\phi_{\,,j} = \frac{\partial\phi}{\partial x^j}, \quad \phi_{\,,jk} = \frac{\partial^2\phi}{\partial x^j \partial x^k},$ etc.

**Divergence.** In Cartesian coordinates the divergence of a vector field $\vec{A}$ is a scalar field and can be represented

$$\nabla \cdot \vec{A} = div\,\vec{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}.$$

Employing the summation convention and index notation, the divergence in Cartesian coordinates can be represented

$$\nabla \cdot \vec{A} = div\,\vec{A} = A_{i,i} = \frac{\partial A_i}{\partial x^i} = \frac{\partial A_1}{\partial x^1} + \frac{\partial A_2}{\partial x^2} + \frac{\partial A_3}{\partial x^3}$$

where $i$ is the dummy summation index.

**Curl.** To represent the vector $\vec{B} = \text{curl}\,\vec{A} = \nabla \times \vec{A}$ in Cartesian coordinates, we note that the index notation focuses attention only on the components of this vector. The components $B_i,\ i = 1, 2, 3$ of $\vec{B}$ can be represented

$$B_i = \widehat{\mathbf{e}}_i \cdot \text{curl}\,\vec{A} = e_{ijk} A_{k,j}, \qquad \text{for} \qquad i, j, k = 1, 2, 3$$

where $e_{ijk}$ is the permutation symbol introduced earlier and $A_{k,j} = \frac{\partial A_k}{\partial x^j}$. To verify this representation of the curl $\vec{A}$ we need only perform the summations indicated by the repeated indices. We have summing on $j$ that

$$B_i = e_{i1k} A_{k,1} + e_{i2k} A_{k,2} + e_{i3k} A_{k,3}.$$

Now summing each term on the repeated index $k$ gives us

$$B_i = e_{i12} A_{2,1} + e_{i13} A_{3,1} + e_{i21} A_{1,2} + e_{i23} A_{3,2} + e_{i31} A_{1,3} + e_{i32} A_{2,3}$$

Here $i$ is a free index which can take on any of the values $1, 2$ or $3$. Consequently, we have

$$\begin{aligned} &\text{For} \quad i = 1, \qquad B_1 = A_{3,2} - A_{2,3} = \frac{\partial A_3}{\partial x^2} - \frac{\partial A_2}{\partial x^3} \\ &\text{For} \quad i = 2, \qquad B_2 = A_{1,3} - A_{3,1} = \frac{\partial A_1}{\partial x^3} - \frac{\partial A_3}{\partial x^1} \\ &\text{For} \quad i = 3, \qquad B_3 = A_{2,1} - A_{1,2} = \frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2} \end{aligned}$$

which verifies the index notation representation of curl $\vec{A}$ in Cartesian coordinates.

**Other Operations.** The following examples illustrate how the index notation can be used to represent additional vector operators in Cartesian coordinates.

1. In index notation the components of the vector $(\vec{B}\cdot\nabla)\vec{A}$ are

$$\{(\vec{B}\cdot\nabla)\vec{A}\}\cdot\widehat{\mathbf{e}}_p = A_{p,q}B_q \qquad p,q=1,2,3$$

This can be verified by performing the indicated summations. We have by summing on the repeated index $q$

$$A_{p,q}B_q = A_{p,1}B_1 + A_{p,2}B_2 + A_{p,3}B_3.$$

The index $p$ is now a free index which can have any of the values $1,2$ or $3$. We have:

$$\begin{aligned}
\text{for}\quad p=1,\qquad A_{1,q}B_q &= A_{1,1}B_1 + A_{1,2}B_2 + A_{1,3}B_3\\
&= \frac{\partial A_1}{\partial x^1}B_1 + \frac{\partial A_1}{\partial x^2}B_2 + \frac{\partial A_1}{\partial x^3}B_3\\
\text{for}\quad p=2,\qquad A_{2,q}B_q &= A_{2,1}B_1 + A_{2,2}B_2 + A_{2,3}B_3\\
&= \frac{\partial A_2}{\partial x^1}B_1 + \frac{\partial A_2}{\partial x^2}B_2 + \frac{\partial A_2}{\partial x^3}B_3\\
\text{for}\quad p=3,\qquad A_{3,q}B_q &= A_{3,1}B_1 + A_{3,2}B_2 + A_{3,3}B_3\\
&= \frac{\partial A_3}{\partial x^1}B_1 + \frac{\partial A_3}{\partial x^2}B_2 + \frac{\partial A_3}{\partial x^3}B_3
\end{aligned}$$

2. The scalar $(\vec{B}\cdot\nabla)\phi$ has the following form when expressed in the index notation:

$$\begin{aligned}
(\vec{B}\cdot\nabla)\phi = B_i\phi_{,i} &= B_1\phi_{,1} + B_2\phi_{,2} + B_3\phi_{,3}\\
&= B_1\frac{\partial\phi}{\partial x^1} + B_2\frac{\partial\phi}{\partial x^2} + B_3\frac{\partial\phi}{\partial x^3}.
\end{aligned}$$

3. The components of the vector $(\vec{B}\times\nabla)\phi$ is expressed in the index notation by

$$\widehat{\mathbf{e}}_i\cdot\left[(\vec{B}\times\nabla)\phi\right] = e_{ijk}B_j\phi_{,k}.$$

This can be verified by performing the indicated summations and is left as an exercise.

4. The scalar $(\vec{B}\times\nabla)\cdot\vec{A}$ may be expressed in the index notation. It has the form

$$(\vec{B}\times\nabla)\cdot\vec{A} = e_{ijk}B_jA_{i,k}.$$

This can also be verified by performing the indicated summations and is left as an exercise.

5. The vector components of $\nabla^2\vec{A}$ in the index notation are represented

$$\widehat{\mathbf{e}}_p\cdot\nabla^2\vec{A} = A_{p,qq}.$$

The proof of this is left as an exercise.

**EXAMPLE 1.1-19.** In Cartesian coordinates prove the vector identity

$$\text{curl } (f\vec{A}) = \nabla \times (f\vec{A}) = (\nabla f) \times \vec{A} + f(\nabla \times \vec{A}).$$

**Solution:** Let $\vec{B} = \text{curl } (f\vec{A})$ and write the components as

$$\begin{aligned} B_i &= e_{ijk}(fA_k)_{,j} \\ &= e_{ijk}\left[fA_{k,j} + f_{,j}A_k\right] \\ &= fe_{ijk}A_{k,j} + e_{ijk}f_{,j}A_k. \end{aligned}$$

This index form can now be expressed in the vector form

$$\vec{B} = \text{curl } (f\vec{A}) = f(\nabla \times \vec{A}) + (\nabla f) \times \vec{A}$$

■

**EXAMPLE 1.1-20.** Prove the vector identity $\nabla \cdot (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$

**Solution:** Let $\vec{A} + \vec{B} = \vec{C}$ and write this vector equation in the index notation as $A_i + B_i = C_i$. We then have

$$\nabla \cdot \vec{C} = C_{i,i} = (A_i + B_i)_{,i} = A_{i,i} + B_{i,i} = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}.$$

■

**EXAMPLE 1.1-21.** In Cartesian coordinates prove the vector identity $(\vec{A} \cdot \nabla)f = \vec{A} \cdot \nabla f$

**Solution:** In the index notation we write

$$\begin{aligned} (\vec{A} \cdot \nabla)f = A_i f_{,i} &= A_1 f_{,1} + A_2 f_{,2} + A_3 f_{,3} \\ &= A_1 \frac{\partial f}{\partial x^1} + A_2 \frac{\partial f}{\partial x^2} + A_3 \frac{\partial f}{\partial x^3} = \vec{A} \cdot \nabla f. \end{aligned}$$

■

**EXAMPLE 1.1-22.** In Cartesian coordinates prove the vector identity

$$\nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$$

**Solution:** The *pth* component of the vector $\nabla \times (\vec{A} \times \vec{B})$ is

$$\begin{aligned} \widehat{\mathbf{e}}_p \cdot [\nabla \times (\vec{A} \times \vec{B})] &= e_{pqk}[e_{kji}A_jB_i]_{,q} \\ &= e_{pqk}e_{kji}A_jB_{i,q} + e_{pqk}e_{kji}A_{j,q}B_i \end{aligned}$$

By applying the $e - \delta$ identity, the above expression simplifies to the desired result. That is,

$$\begin{aligned} \widehat{\mathbf{e}}_p \cdot [\nabla \times (\vec{A} \times \vec{B})] &= (\delta_{pj}\delta_{qi} - \delta_{pi}\delta_{qj})A_jB_{i,q} + (\delta_{pj}\delta_{qi} - \delta_{pi}\delta_{qj})A_{j,q}B_i \\ &= A_pB_{i,i} - A_qB_{p,q} + A_{p,q}B_q - A_{q,q}B_p \end{aligned}$$

In vector form this is expressed

$$\nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \cdot \vec{B}) - (\vec{A} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla)\vec{A} - \vec{B}(\nabla \cdot \vec{A})$$

■

**EXAMPLE 1.1-23.** In Cartesian coordinates prove the vector identity $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$
**Solution:** We have for the $ith$ component of $\nabla \times \vec{A}$ is given by $\widehat{\mathbf{e}}_i \cdot [\nabla \times \vec{A}] = e_{ijk} A_{k,j}$ and consequently the $pth$ component of $\nabla \times (\nabla \times \vec{A})$ is

$$\begin{aligned} \widehat{\mathbf{e}}_p \cdot [\nabla \times (\nabla \times \vec{A})] &= e_{pqr}[e_{rjk} A_{k,j}]_{,q} \\ &= e_{pqr} e_{rjk} A_{k,jq}. \end{aligned}$$

The $e - \delta$ identity produces

$$\begin{aligned} \widehat{\mathbf{e}}_p \cdot [\nabla \times (\nabla \times \vec{A})] &= (\delta_{pj}\delta_{qk} - \delta_{pk}\delta_{qj}) A_{k,jq} \\ &= A_{k,pk} - A_{p,qq}. \end{aligned}$$

Expressing this result in vector form we have $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$.

∎

**Indicial Form of Integral Theorems**

The divergence theorem, in both vector and indicial notation, can be written

$$\iiint_V \operatorname{div} \cdot \vec{F}\, d\tau = \iint_S \vec{F} \cdot \widehat{\mathbf{n}}\, d\sigma \qquad \int_V F_{i,i}\, d\tau = \int_S F_i n_i\, d\sigma \qquad i = 1, 2, 3 \tag{1.1.16}$$

where $n_i$ are the direction cosines of the unit exterior normal to the surface, $d\tau$ is a volume element and $d\sigma$ is an element of surface area. Note that in using the indicial notation the volume and surface integrals are to be extended over the range specified by the indices. This suggests that the divergence theorem can be applied to vectors in $n-$dimensional spaces.

The vector form and indicial notation for the Stokes theorem are

$$\iint_S (\nabla \times \vec{F}) \cdot \widehat{\mathbf{n}}\, d\sigma = \int_C \vec{F} \cdot d\vec{r} \qquad \int_S e_{ijk} F_{k,j} n_i\, d\sigma = \int_C F_i\, dx^i \qquad i, j, k = 1, 2, 3 \tag{1.1.17}$$

and the Green's theorem in the plane, which is a special case of the Stoke's theorem, can be expressed

$$\iint \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \int_C F_1\, dx + F_2\, dy \qquad \int_S e_{3jk} F_{k,j}\, dS = \int_C F_i\, dx^i \quad i, j, k = 1, 2 \tag{1.1.18}$$

Other forms of the above integral theorems are

$$\iiint_V \nabla\phi\, d\tau = \iint_S \phi\, \widehat{\mathbf{n}}\, d\sigma$$

obtained from the divergence theorem by letting $\vec{F} = \phi \vec{C}$ where $\vec{C}$ is a constant vector. By replacing $\vec{F}$ by $\vec{F} \times \vec{C}$ in the divergence theorem one can derive

$$\iiint_V \left( \nabla \times \vec{F} \right) d\tau = -\iint_S \vec{F} \times \vec{n}\, d\sigma.$$

In the divergence theorem make the substitution $\vec{F} = \phi \nabla \psi$ to obtain

$$\iiint_V \left[ (\phi \nabla^2 \psi + (\nabla\phi) \cdot (\nabla\psi) \right] d\tau = \iint_S (\phi \nabla \psi) \cdot \widehat{\mathbf{n}}\, d\sigma.$$

The Green's identity

$$\iiint_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)\, d\tau = \iint_S (\phi\nabla\psi - \psi\nabla\phi)\cdot\widehat{\mathbf{n}}\, d\sigma$$

is obtained by first letting $\vec{F} = \phi\nabla\psi$ in the divergence theorem and then letting $\vec{F} = \psi\nabla\phi$ in the divergence theorem and then subtracting the results.

**Determinants, Cofactors**

For $A = (a_{ij}),\ i,j = 1,\ldots,n$ an $n\times n$ matrix, the determinant of $A$ can be written as

$$\det A = |A| = e_{i_1 i_2 i_3 \ldots i_n} a_{1i_1} a_{2i_2} a_{3i_3} \ldots a_{ni_n}.$$

This gives a summation of the $n!$ permutations of products formed from the elements of the matrix $A$. The result is a single number called the determinant of $A$.

**EXAMPLE 1.1-24.** In the case $n = 2$ we have

$$\begin{aligned} |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} &= e_{nm} a_{1n} a_{2m} \\ &= e_{1m} a_{11} a_{2m} + e_{2m} a_{12} a_{2m} \\ &= e_{12} a_{11} a_{22} + e_{21} a_{12} a_{21} \\ &= a_{11}a_{22} - a_{12}a_{21} \end{aligned}$$

■

**EXAMPLE 1.1-25.** In the case $n = 3$ we can use either of the notations

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \text{or} \qquad A = \begin{pmatrix} a_1^1 & a_2^1 & a_3^1 \\ a_1^2 & a_2^2 & a_3^2 \\ a_1^3 & a_2^3 & a_3^3 \end{pmatrix}$$

and represent the determinant of $A$ in any of the forms

$$\begin{aligned} \det A &= e_{ijk} a_{1i} a_{2j} a_{3k} \\ \det A &= e_{ijk} a_{i1} a_{j2} a_{k3} \\ \det A &= e_{ijk} a_1^i a_2^j a_3^k \\ \det A &= e_{ijk} a_i^1 a_j^2 a_k^3. \end{aligned}$$

These represent row and column expansions of the determinant.

An important identity results if we examine the quantity $B_{rst} = e_{ijk} a_r^i a_s^j a_t^k$. It is an easy exercise to change the dummy summation indices and rearrange terms in this expression. For example,

$$B_{rst} = e_{ijk} a_r^i a_s^j a_t^k = e_{kji} a_r^k a_s^j a_t^i = e_{kji} a_t^i a_s^j a_r^k = -e_{ijk} a_t^i a_s^j a_r^k = -B_{tsr},$$

and by considering other permutations of the indices, one can establish that $B_{rst}$ is completely skew-symmetric. In the exercises it is shown that any third order completely skew-symmetric system satisfies $B_{rst} = B_{123} e_{rst}$. But $B_{123} = \det A$ and so we arrive at the identity

$$B_{rst} = e_{ijk} a_r^i a_s^j a_t^k = |A| e_{rst}.$$

Other forms of this identity are

$$e^{ijk} a_i^r a_j^s a_k^t = |A| e^{rst} \quad \text{and} \quad e_{ijk} a_{ir} a_{js} a_{kt} = |A| e_{rst}. \tag{1.1.19}$$

■

Consider the representation of the determinant

$$|A| = \begin{vmatrix} a_1^1 & a_2^1 & a_3^1 \\ a_1^2 & a_2^2 & a_3^2 \\ a_1^3 & a_2^3 & a_3^3 \end{vmatrix}$$

by use of the indicial notation. By column expansions, this determinant can be represented

$$|A| = e_{rst} a_1^r a_2^s a_3^t \tag{1.1.20}$$

and if one uses row expansions the determinant can be expressed as

$$|A| = e^{ijk} a_i^1 a_j^2 a_k^3. \tag{1.1.21}$$

Define $A_m^i$ as the cofactor of the element $a_i^m$ in the determinant $|A|$. From the equation (1.1.20) the cofactor of $a_1^r$ is obtained by deleting this element and we find

$$A_r^1 = e_{rst} a_2^s a_3^t. \tag{1.1.22}$$

The result (1.1.20) can then be expressed in the form

$$|A| = a_1^r A_r^1 = a_1^1 A_1^1 + a_1^2 A_2^1 + a_1^3 A_3^1. \tag{1.1.23}$$

That is, the determinant $|A|$ is obtained by multiplying each element in the first column by its corresponding cofactor and summing the result. Observe also that from the equation (1.1.20) we find the additional cofactors

$$A_s^2 = e_{rst} a_1^r a_3^t \qquad \text{and} \qquad A_t^3 = e_{rst} a_1^r a_2^s. \tag{1.1.24}$$

Hence, the equation (1.1.20) can also be expressed in one of the forms

$$|A| = a_2^s A_s^2 = a_2^1 A_1^2 + a_2^2 A_2^2 + a_2^3 A_3^2$$
$$|A| = a_3^t A_t^3 = a_3^1 A_1^3 + a_3^2 A_2^3 + a_3^3 A_3^3$$

The results from equations (1.1.22) and (1.1.24) can be written in a slightly different form with the indicial notation. From the notation for a generalized Kronecker delta defined by

$$e^{ijk} e_{lmn} = \delta_{lmn}^{ijk},$$

the above cofactors can be written in the form

$$A_r^1 = e^{123} e_{rst} a_2^s a_3^t = \frac{1}{2!} e^{1jk} e_{rst} a_j^s a_k^t = \frac{1}{2!} \delta_{rst}^{1jk} a_j^s a_k^t$$
$$A_r^2 = e^{123} e_{srt} a_1^s a_3^t = \frac{1}{2!} e^{2jk} e_{rst} a_j^s a_k^t = \frac{1}{2!} \delta_{rst}^{2jk} a_j^s a_k^t$$
$$A_r^3 = e^{123} e_{tsr} a_1^t a_2^s = \frac{1}{2!} e^{3jk} e_{rst} a_j^s a_k^t = \frac{1}{2!} \delta_{rst}^{3jk} a_j^s a_k^t.$$

These cofactors are then combined into the single equation

$$A^i_r = \frac{1}{2!}\delta^{ijk}_{rst} a^s_j a^t_k \tag{1.1.25}$$

which represents the cofactor of $a^r_i$. When the elements from any row (or column) are multiplied by their corresponding cofactors, and the results summed, we obtain the value of the determinant. Whenever the elements from any row (or column) are multiplied by the cofactor elements from a different row (or column), and the results summed, we get zero. This can be illustrated by considering the summation

$$\begin{aligned} a^m_r A^i_m &= \frac{1}{2!}\delta^{ijk}_{mst} a^s_j a^t_k a^m_r = \frac{1}{2!} e^{ijk} e_{mst} a^m_r a^s_j a^t_k \\ &= \frac{1}{2!} e^{ijk} e_{rjk} |A| = \frac{1}{2!}\delta^{ijk}_{rjk}|A| = \delta^i_r |A| \end{aligned}$$

Here we have used the $e - \delta$ identity to obtain

$$\delta^{ijk}_{rjk} = e^{ijk} e_{rjk} = e^{jik} e_{jrk} = \delta^i_r \delta^k_k - \delta^i_k \delta^k_r = 3\delta^i_r - \delta^i_r = 2\delta^i_r$$

which was used to simplify the above result.

As an exercise one can show that an alternate form of the above summation of elements by its cofactors is

$$a^r_m A^m_i = |A|\delta^r_i.$$

**EXAMPLE 1.1-26.** In N-dimensions the quantity $\delta^{j_1 j_2 \ldots j_N}_{k_1 k_2 \ldots k_N}$ is called a generalized Kronecker delta. It can be defined in terms of permutation symbols as

$$e^{j_1 j_2 \ldots j_N} e_{k_1 k_2 \ldots k_N} = \delta^{j_1 j_2 \ldots j_N}_{k_1 k_2 \ldots k_N} \tag{1.1.26}$$

Observe that

$$\delta^{j_1 j_2 \ldots j_N}_{k_1 k_2 \ldots k_N} e^{k_1 k_2 \ldots k_N} = (N!)\, e^{j_1 j_2 \ldots j_N}$$

This follows because $e^{k_1 k_2 \ldots k_N}$ is skew-symmetric in all pairs of its superscripts. The left-hand side denotes a summation of $N!$ terms. The first term in the summation has superscripts $j_1 j_2 \ldots j_N$ and all other terms have superscripts which are some permutation of this ordering with minus signs associated with those terms having an odd permutation. Because $e^{j_1 j_2 \ldots j_N}$ is completely skew-symmetric we find that all terms in the summation have the value $+e^{j_1 j_2 \ldots j_N}$. We thus obtain $N!$ of these terms.

■

## EXERCISE 1.1

► **1.** Simplify each of the following by employing the summation property of the Kronecker delta. Perform sums on the summation indices only if your are unsure of the result.

$(a)\quad e_{ijk}\delta_{kn}$ $\qquad(c)\quad e_{ijk}\delta_{is}\delta_{jm}\delta_{kn}$ $\qquad(e)\quad \delta_{ij}\delta_{jn}$

$(b)\quad e_{ijk}\delta_{is}\delta_{jm}$ $\qquad(d)\quad a_{ij}\delta_{in}$ $\qquad(f)\quad \delta_{ij}\delta_{jn}\delta_{ni}$

► **2.** Simplify and perform the indicated summations over the range $1,2,3$

$(a)\quad \delta_{ii}$ $\qquad(c)\quad e_{ijk}A_iA_jA_k$ $\qquad(e)\quad e_{ijk}\delta_{jk}$

$(b)\quad \delta_{ij}\delta_{ij}$ $\qquad(d)\quad e_{ijk}e_{ijk}$ $\qquad(f)\quad A_iB_j\delta_{ji} - B_mA_n\delta_{mn}$

► **3.** Express each of the following in index notation. Be careful of the notation you use. Note that $\vec{A} = A_i$ is an incorrect notation because a vector can not equal a scalar. The notation $\vec{A}\cdot\widehat{\mathbf{e}}_i = A_i$ should be used to express the $ith$ component of a vector.

$(a)\quad \vec{A}\cdot(\vec{B}\times\vec{C})$ $\qquad(c)\quad \vec{B}(\vec{A}\cdot\vec{C})$

$(b)\quad \vec{A}\times(\vec{B}\times\vec{C})$ $\qquad(d)\quad \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

► **4.** Show the $e$ permutation symbol satisfies: $(a)\quad e_{ijk} = e_{jki} = e_{kij}$ $\quad(b)\quad e_{ijk} = -e_{jik} = -e_{ikj} = -e_{kji}$

► **5.** Use index notation to verify the vector identity $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

► **6.** Let $y_i = a_{ij}x_j$ and $x_m = a_{im}z_i$ where the range of the indices is $1,2$

$(a)$ Solve for $y_i$ in terms of $z_i$ using the indicial notation and check your result to be sure that no index repeats itself more than twice.

$(b)$ Perform the indicated summations and write out expressions for $y_1, y_2$ in terms of $z_1, z_2$

$(c)$ Express the above equations in matrix form. Expand the matrix equations and check the solution obtained in part (b).

► **7.** Use the $e-\delta$ identity to simplify $(a)\quad e_{ijk}e_{jik}$ $\qquad(b)\quad e_{ijk}e_{jki}$

► **8.** Prove the following vector identities:

$(a)\quad \vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$ triple scalar product

$(b)\quad (\vec{A}\times\vec{B})\times\vec{C} = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{A}(\vec{B}\cdot\vec{C})$

► **9.** Prove the following vector identities:

$(a)\quad (\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})$

$(b)\quad \vec{A}\times(\vec{B}\times\vec{C}) + \vec{B}\times(\vec{C}\times\vec{A}) + \vec{C}\times(\vec{A}\times\vec{B}) = \vec{0}$

$(c)\quad (\vec{A}\times\vec{B})\times(\vec{C}\times\vec{D}) = \vec{B}(\vec{A}\cdot\vec{C}\times\vec{D}) - \vec{A}(\vec{B}\cdot\vec{C}\times\vec{D})$

▶ **10.** For $\vec{A} = (1, -1, 0)$ and $\vec{B} = (4, -3, 2)$ find using the index notation,

$(a)$ $C_i = e_{ijk} A_j B_k, \quad i = 1, 2, 3$

$(b)$ $A_i B_i$

$(c)$ What do the results in (a) and (b) represent?

▶ **11.** Represent the differential equations $\dfrac{dy_1}{dt} = a_{11} y_1 + a_{12} y_2$ and $\dfrac{dy_2}{dt} = a_{21} y_1 + a_{22} y_2$ using the index notation.

▶ **12.**

Let $\Phi = \Phi(r, \theta)$ where $r, \theta$ are polar coordinates related to Cartesian coordinates $(x, y)$ by the transformation equations $x = r\cos\theta$ and $y = r\sin\theta$.

(a) Find the partial derivatives $\dfrac{\partial \Phi}{\partial y}$, and $\dfrac{\partial^2 \Phi}{\partial y^2}$

(b) Combine the result in part (a) with the result from EXAMPLE 1.1-18 to calculate the Laplacian

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}$$

in polar coordinates.

▶ **13.** (Index notation) Let $a_{11} = 3, \quad a_{12} = 4, \quad a_{21} = 5, \quad a_{22} = 6.$
Calculate the quantity $C = a_{ij} a_{ij}$, $i, j = 1, 2.$

▶ **14.** Show the moments of inertia $I_{ij}$ defined by

$$I_{11} = \iiint_R (y^2 + z^2)\rho(x, y, z)\, d\tau \qquad I_{23} = I_{32} = -\iiint_R yz\rho(x, y, z)\, d\tau$$

$$I_{22} = \iiint_R (x^2 + z^2)\rho(x, y, z)\, d\tau \qquad I_{12} = I_{21} = -\iiint_R xy\rho(x, y, z)\, d\tau$$

$$I_{33} = \iiint_R (x^2 + y^2)\rho(x, y, z)\, d\tau \qquad I_{13} = I_{31} = -\iiint_R xz\rho(x, y, z)\, d\tau,$$

can be represented in the index notation as $I_{ij} = \iiint_R \left(x^m x^m \delta_{ij} - x^i x^j\right) \rho\, d\tau$, where $\rho$ is the density, $x^1 = x,\ x^2 = y,\ x^3 = z$ and $d\tau = dxdydz$ is an element of volume.

▶ **15.** Determine if the following relation is true or false. Justify your answer.

$$\widehat{\mathbf{e}}_i \cdot (\widehat{\mathbf{e}}_j \times \widehat{\mathbf{e}}_k) = (\widehat{\mathbf{e}}_i \times \widehat{\mathbf{e}}_j) \cdot \widehat{\mathbf{e}}_k = e_{ijk}, \quad i, j, k = 1, 2, 3.$$

Hint: Let $\widehat{\mathbf{e}}_m = (\delta_{1m}, \delta_{2m}, \delta_{3m})$.

▶ **16.** Without substituting values for $i, l = 1, 2, 3$ calculate all nine terms of the given quantities

$(a)$ $B^{il} = (\delta^i_j A_k + \delta^i_k A_j) e^{jkl}$ $\qquad$ $(b)$ $A_{il} = (\delta^m_i B^k + \delta^k_i B^m) e_{mlk}$

▶ **17.** Let $A_{mn} x^m y^n = 0$ for arbitrary $x^i$ and $y^i$, $\quad i = 1, 2, 3$, and show that $A_{ij} = 0$ for all values of $i, j$.

► **18.**

(a) For $a_{mn}, m, n = 1, 2, 3$ skew-symmetric, show that $a_{mn}x^m x^n = 0$.

(b) Let $a_{mn}x^m x^n = 0, \quad m, n = 1, 2, 3$ for all values of $x^i, i = 1, 2, 3$ and show that $a_{mn}$ must be skew-symmetric.

► **19.** Let $A$ and $B$ denote $3 \times 3$ matrices with elements $a_{ij}$ and $b_{ij}$ respectively. Show that if $C = AB$ is a matrix product, then $det(C) = det(A) \cdot det(B)$.

Hint: Use the result from example 1.1-9.

► **20.**

(a) Let $u^1, u^2, u^3$ be functions of the variables $s^1, s^2, s^3$. Further, assume that $s^1, s^2, s^3$ are in turn each functions of the variables $x^1, x^2, x^3$. Let $\left|\frac{\partial u^m}{\partial x^n}\right| = \frac{\partial(u^1, u^2, u^3)}{\partial(x^1, x^2, x^3)}$ denote the Jacobian of the $u's$ with respect to the $x's$. Show that

$$\left|\frac{\partial u^i}{\partial x^m}\right| = \left|\frac{\partial u^i}{\partial s^j}\frac{\partial s^j}{\partial x^m}\right| = \left|\frac{\partial u^i}{\partial s^j}\right| \cdot \left|\frac{\partial s^j}{\partial x^m}\right|.$$

(b) Note that $\frac{\partial x^i}{\partial \bar{x}^j}\frac{\partial \bar{x}^j}{\partial x^m} = \frac{\partial x^i}{\partial x^m} = \delta^i_m$ and show that $J(\frac{x}{\bar{x}}) \cdot J(\frac{\bar{x}}{x}) = 1$, where $J(\frac{x}{\bar{x}})$ is the Jacobian determinant of the transformation (1.1.7).

► **21.** A third order system $a_{\ell mn}$ with $\ell, m, n = 1, 2, 3$ is said to be symmetric in two of its subscripts if the components are unaltered when these subscripts are interchanged. When $a_{\ell mn}$ is completely symmetric then $a_{\ell mn} = a_{m\ell n} = a_{\ell nm} = a_{mn\ell} = a_{nm\ell} = a_{n\ell m}$. Whenever this third order system is completely symmetric, then: (i) How many components are there? (ii) How many of these components are distinct?

Hint: Consider the three cases (i) $\ell = m = n$ (ii) $\ell = m \neq n$ (iii) $\ell \neq m \neq n$.

► **22.** A third order system $b_{\ell mn}$ with $\ell, m, n = 1, 2, 3$ is said to be skew-symmetric in two of its subscripts if the components change sign when the subscripts are interchanged. A completely skew-symmetric third order system satisfies $b_{\ell mn} = -b_{m\ell n} = b_{mn\ell} = -b_{nm\ell} = b_{n\ell m} = -b_{\ell nm}$. (i) How many components does a completely skew-symmetric system have? (ii) How many of these components are zero? (iii) How many components can be different from zero? (iv) Show that there is one distinct component $b_{123}$ and that $b_{\ell mn} = e_{\ell mn} b_{123}$.

Hint: Consider the three cases (i) $\ell = m = n$ (ii) $\ell = m \neq n$ (iii) $\ell \neq m \neq n$.

► **23.** Let $i, j, k = 1, 2, 3$ and assume that $e_{ijk}\sigma_{jk} = 0$ for all values of $i$. What does this equation tell you about the values $\sigma_{ij}, \quad i, j = 1, 2, 3$?

► **24.** Assume that $A_{mn}$ and $B_{mn}$ are symmetric for $m, n = 1, 2, 3$. Let $A_{mn}x^m x^n = B_{mn}x^m x^n$ for arbitrary values of $x^i, i = 1, 2, 3$, and show that $A_{ij} = B_{ij}$ for all values of $i$ and $j$.

► **25.** Assume $B_{mn}$ is symmetric and $B_{mn}x^m x^n = 0$ for arbitrary values of $x^i, i = 1, 2, 3$, show that $B_{ij} = 0$.

▶ **26.** (**Generalized Kronecker delta**) Define the generalized Kronecker delta as the $n \times n$ determinant

$$\delta^{ij\ldots k}_{mn\ldots p} = \begin{vmatrix} \delta^i_m & \delta^i_n & \cdots & \delta^i_p \\ \delta^j_m & \delta^j_n & \cdots & \delta^j_p \\ \vdots & \vdots & \ddots & \vdots \\ \delta^k_m & \delta^k_n & \cdots & \delta^k_p \end{vmatrix} \quad \text{where } \delta^r_s \text{ is the Kronecker delta.}$$

$(a)$ Show $e_{ijk} = \delta^{123}_{ijk}$

$(b)$ Show $e^{ijk} = \delta^{ijk}_{123}$

$(c)$ Show $\delta^{ij}_{mn} = e^{ij} e_{mn}$

$(d)$ Define $\delta^{rs}_{mn} = \delta^{rsp}_{mnp}$ (summation on $p$)

and show $\delta^{rs}_{mn} = \delta^r_m \delta^s_n - \delta^r_n \delta^s_m$

Note that by combining the above result with the result from part (c)

we obtain the two dimensional form of the $e - \delta$ identity $e^{rs} e_{mn} = \delta^r_m \delta^s_n - \delta^r_n \delta^s_m$.

$(e)$ Define $\delta^r_m = \frac{1}{2}\delta^{rn}_{mn}$ (summation on $n$) and show $\delta^{rst}_{pst} = 2\delta^r_p$

$(f)$ Show $\delta^{rst}_{rst} = 3!$

▶ **27.** Let $A^i_r$ denote the cofactor of $a^r_i$ in the determinant $\begin{vmatrix} a^1_1 & a^1_2 & a^1_3 \\ a^2_1 & a^2_2 & a^2_3 \\ a^3_1 & a^3_2 & a^3_3 \end{vmatrix}$ as given by equation (1.1.25).

$(a)$ Show $e^{rst} A^i_r = e^{ijk} a^s_j a^t_k$ $\qquad$ $(b)$ Show $e_{rst} A^r_i = e_{ijk} a^j_s a^k_t$

▶ **28.** (a) Show that if $A_{ijk} = A_{jik}, i, j, k = 1, 2, 3$ there is a total of 27 elements, but only 18 are distinct. (b) Show that for $i, j, k = 1, 2, \ldots, N$ there are $N^3$ elements, but only $N^2(N+1)/2$ are distinct.

▶ **29.** Let $a_{ij} = B_i B_j$ for $i, j = 1, 2, 3$ where $B_1, B_2, B_3$ are arbitrary constants. Calculate $\det(a_{ij}) = |A|$.

▶ **30.**

$(a)$ For $A = (a_{ij}),\ i, j = 1, 2, 3,$ show $|A| = e_{ijk} a_{i1} a_{j2} a_{k3}$.

$(b)$ For $A = (a^i_j),\ i, j = 1, 2, 3,$ show $|A| = e_{ijk} a^i_1 a^j_2 a^k_3$.

$(c)$ For $A = (a^i_j),\ i, j = 1, 2, 3,$ show $|A| = e^{ijk} a^1_i a^2_j a^3_k$.

$(d)$ For $I = (\delta^i_j),\ i, j = 1, 2, 3,$ show $|I| = 1$.

▶ **31.** Let $|A| = e_{ijk} a_{i1} a_{j2} a_{k3}$ and define $A_{im}$ as the cofactor of $a_{im}$. Show the determinant can be expressed in any of the forms:

$(a)$ $|A| = A_{i1} a_{i1}$ where $A_{i1} = e_{ijk} a_{j2} a_{k3}$

$(b)$ $|A| = A_{j2} a_{j2}$ where $A_{i2} = e_{jik} a_{j1} a_{k3}$

$(c)$ $|A| = A_{k3} a_{k3}$ where $A_{i3} = e_{jki} a_{j1} a_{k2}$

▶ **32.** Show the results in problem 31 can be written in the forms:

$$A_{i1} = \frac{1}{2!}e_{1st}e_{ijk}a_{js}a_{kt}, \quad A_{i2} = \frac{1}{2!}e_{2st}e_{ijk}a_{js}a_{kt}, \quad A_{i3} = \frac{1}{2!}e_{3st}e_{ijk}a_{js}a_{kt}, \quad \text{or} \quad A_{im} = \frac{1}{2!}e_{mst}e_{ijk}a_{js}a_{kt}$$

▶ **33.** Use the results in problems 31 and 32 to prove that $a_{pm}A_{im} = |A|\delta_{ip}$.

▶ **34.** Let $(a_{ij}) = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & 3 & 2 \end{pmatrix}$ and calculate $C = a_{ij}a_{ij},\ i,j = 1,2,3$.

▶ **35.** Let

$$a_{111} = -1, \quad a_{112} = 3, \quad a_{121} = 4, \quad a_{122} = 2$$
$$a_{211} = 1, \quad a_{212} = 5, \quad a_{221} = 2, \quad a_{222} = -2$$

and calculate the quantity $C = a_{ijk}a_{ijk},\ i,j,k = 1,2$.

▶ **36.** Let

$$a_{1111} = 2, \quad a_{1112} = 1, \quad a_{1121} = 3, \quad a_{1122} = 1$$
$$a_{1211} = 5, \quad a_{1212} = -2, \quad a_{1221} = 4, \quad a_{1222} = -2$$
$$a_{2111} = 1, \quad a_{2112} = 0, \quad a_{2121} = -2, \quad a_{2122} = -1$$
$$a_{2211} = -2, \quad a_{2212} = 1, \quad a_{2221} = 2, \quad a_{2222} = 2$$

and calculate the quantity $C = a_{ijkl}a_{ijkl},\ i,j,k,l = 1,2$.

▶ **37.** Simplify the expressions:

$(a)\quad (A_{ijkl} + A_{jkli} + A_{klij} + A_{lijk})x_i x_j x_k x_l$

$(b)\quad (P_{ijk} + P_{jki} + P_{kij})x^i x^j x^k$

$(c)\quad \dfrac{\partial x^i}{\partial x^j}$

$(d)\quad a_{ij}\dfrac{\partial^2 x^i}{\partial \overline{x}^t \partial \overline{x}^s}\dfrac{\partial x^j}{\partial \overline{x}^r} - a_{mi}\dfrac{\partial^2 x^m}{\partial \overline{x}^s \partial \overline{x}^t}\dfrac{\partial x^i}{\partial \overline{x}^r}$

▶ **38.** Let $g$ denote the determinant of the matrix having the components $g_{ij},\ i,j = 1,2,3$. Show that

$$(a)\quad g\,e_{rst} = \begin{vmatrix} g_{1r} & g_{1s} & g_{1t} \\ g_{2r} & g_{2s} & g_{2t} \\ g_{3r} & g_{3s} & g_{3t} \end{vmatrix} \qquad (b)\quad g\,e_{rst}e_{ijk} = \begin{vmatrix} g_{ir} & g_{is} & g_{it} \\ g_{jr} & g_{js} & g_{jt} \\ g_{kr} & g_{ks} & g_{kt} \end{vmatrix}$$

▶ **39.** Show that $e^{ijk}e_{mnp} = \delta^{ijk}_{mnp} = \begin{vmatrix} \delta^i_m & \delta^i_n & \delta^i_p \\ \delta^j_m & \delta^j_n & \delta^j_p \\ \delta^k_m & \delta^k_n & \delta^k_p \end{vmatrix}$

▶ **40.** Show that $e^{ijk}e_{mnp}A^{mnp} = A^{ijk} - A^{ikj} + A^{kij} - A^{jik} + A^{jki} - A^{kji}$

Hint: Use the results from problem 39.

▶ **41.** Show that

$(a)\quad e^{ij}e_{ij} = 2!$

$(b)\quad e^{ijk}e_{ijk} = 3!$

$(c)\quad e^{ijkl}e_{ijkl} = 4!$

$(d)\quad$ Guess at the result $\quad e^{i_1 i_2 \ldots i_n}\, e_{i_1 i_2 \ldots i_n}$

▶ **42.** Determine if the following statement is true or false. Justify your answer. $e_{ijk}A_iB_jC_k = e_{ijk}A_jB_kC_i$.

▶ **43.** Let $a_{ij}$, $i,j = 1,2$ denote the components of a $2\times 2$ matrix $A$, which are functions of time $t$.

(a) Expand both $|A| = e_{ij}a_{i1}a_{j2}$ and $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ to verify that these representations are the same.

(b) Verify the equivalence of the derivative relations

$$\frac{d|A|}{dt} = e_{ij}\frac{da_{i1}}{dt}a_{j2} + e_{ij}a_{i1}\frac{da_{j2}}{dt} \quad \text{and} \quad \frac{d|A|}{dt} = \begin{vmatrix} \frac{da_{11}}{dt} & \frac{da_{12}}{dt} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ \frac{da_{21}}{dt} & \frac{da_{22}}{dt} \end{vmatrix}$$

(c) Let $a_{ij}$, $i,j = 1,2,3$ denote the components of a $3\times 3$ matrix $A$, which are functions of time $t$. Develop appropriate relations, expand them and verify, similar to parts (a) and (b) above, the representation of a determinant and its derivative.

▶ **44.** For $f = f(x^1, x^2, x^3)$ and $\phi = \phi(f)$ differentiable scalar functions, use the indicial notation to find a formula to calculate grad $\phi$.

▶ **45.** Use the indicial notation to prove $(a)\quad \nabla\times\nabla\phi = \vec{0} \qquad\qquad (b)\quad \nabla\cdot\nabla\times\vec{A} = 0$

▶ **46.** If $A_{ij}$ is symmetric and $B_{ij}$ is skew-symmetric, $i,j = 1,2,3$, then calculate $C = A_{ij}B_{ij}$.

▶ **47.** Assume $\overline{A}_{ij} = \overline{A}_{ij}(\overline{x}^1, \overline{x}^2, \overline{x}^3)$ and $A_{ij} = A_{ij}(x^1, x^2, x^3)$ for $i,j = 1,2,3$ are related by the expression $\overline{A}_{mn} = A_{ij}\dfrac{\partial x^i}{\partial \overline{x}^m}\dfrac{\partial x^j}{\partial \overline{x}^n}$. Calculate the derivative $\dfrac{\partial \overline{A}_{mn}}{\partial \overline{x}^k}$.

▶ **48.** Prove that if any two rows (or two columns) of a matrix are interchanged, then the value of the determinant of the matrix is multiplied by minus one. Construct your proof using $3\times 3$ matrices.

▶ **49.** Prove that if two rows (or columns) of a matrix are proportional, then the value of the determinant of the matrix is zero. Construct your proof using $3\times 3$ matrices.

▶ **50.** Prove that if a row (or column) of a matrix is altered by adding some constant multiple of some other row (or column), then the value of the determinant of the matrix remains unchanged. Construct your proof using $3\times 3$ matrices.

▶ **51.** Simplify the expression $\phi = e_{ijk}e_{\ell mn}A_{i\ell}A_{jm}A_{kn}$.

▶ **52.** Let $A_{ijk}$ denote a third order system where $i,j,k = 1,2$. (a) How many components does this system have? (b) Let $A_{ijk}$ be skew-symmetric in the last pair of indices, how many independent components does the system have?

▶ **53.** Let $A_{ijk}$ denote a third order system where $i,j,k = 1,2,3$. (a) How many components does this system have? (b) In addition let $A_{ijk} = A_{jik}$ and $A_{ikj} = -A_{ijk}$ and determine the number of distinct nonzero components for $A_{ijk}$.

▶ **54.** Show that every second order system $T_{ij}$ can be expressed as the sum of a symmetric system $A_{ij}$ and skew-symmetric system $B_{ij}$. Find $A_{ij}$ and $B_{ij}$ in terms of the components of $T_{ij}$.

▶ **55.** Consider the system $A_{ijk}, \quad i,j,k = 1,2,3,4$.

(a) How many components does this system have?

(b) Assume $A_{ijk}$ is skew-symmetric in the last pair of indices, how many independent components does this system have?

(c) Assume that in addition to being skew-symmetric in the last pair of indices, $A_{ijk} + A_{jki} + A_{kij} = 0$ is satisfied for all values of $i, j$, and $k$, then how many independent components does the system have?

▶ **56.** (a) Write the equation of a line $\vec{r} = \vec{r}_0 + t\,\vec{A}$ in indicial form. (b) Write the equation of the plane $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$ in indicial form. (c) Write the equation of a general line in scalar form. (d) Write the equation of a plane in scalar form. (e) Find the equation of the line defined by the intersection of the planes $2x + 3y + 6z = 12$ and $6x + 3y + z = 6$. (f) Find the equation of the plane through the points $(5,3,2), (3,1,5), (1,3,3)$. Find also the normal to this plane.

▶ **57.** The angle $0 \le \theta \le \pi$ between two skew lines in space is defined as the angle between their direction vectors when these vectors are placed at the origin. Show that for two lines with direction numbers $a_i$ and $b_i$ $i = 1,2,3$, the cosine of the angle between these lines satisfies

$$\cos\theta = \frac{a_i b_i}{\sqrt{a_i a_i}\sqrt{b_i b_i}}$$

▶ **58.** Let $a_{ij} = -a_{ji}$ for $i,j = 1,2,\ldots,N$ and prove that for $N$ odd $det(a_{ij}) = 0$.

▶ **59.** Let $\lambda = A_{ij} x_i x_j$ where $A_{ij} = A_{ji}$ and calculate (a) $\dfrac{\partial \lambda}{\partial x_m}$ (b) $\dfrac{\partial^2 \lambda}{\partial x_m \partial x_k}$

▶ **60.** Given an arbitrary nonzero vector $U_k$, $k = 1,2,3$, define the matrix elements $a_{ij} = e_{ijk} U_k$, where $e_{ijk}$ is the e-permutation symbol. Determine if $a_{ij}$ is symmetric or skew-symmetric. Suppose $U_k$ is defined by the above equation for arbitrary nonzero $a_{ij}$, then solve for $U_k$ in terms of the $a_{ij}$.

▶ **61.** If $A_{ij} = A_i B_j \neq 0$ for all $i,j$ values and $A_{ij} = A_{ji}$ for $i,j = 1,2,\ldots,N$, show that $A_{ij} = \lambda B_i B_j$ where $\lambda$ is a constant. State what $\lambda$ is.

▶ **62.** Assume that $A_{ijkm}$, with $i,j,k,m = 1,2,3$, is completely skew-symmetric. How many independent components does this quantity have?

▶ **63.** Consider $R_{ijkm}, i,j,k,m = 1,2,3,4$. (a) How many components does this quantity have? (b) If $R_{ijkm} = -R_{ijmk} = -R_{jikm}$ then how many independent components does $R_{ijkm}$ have? (c) If in addition $R_{ijkm} = R_{kmij}$ determine the number of independent components.

▶ **64.** Let $x_i = a_{ij} \bar{x}_j, \quad i,j = 1,2,3$ denote a change of variables from a barred system of coordinates to an unbarred system of coordinates and assume that $\bar{A}_i = a_{ij} A_j$ where $a_{ij}$ are constants, $\bar{A}_i$ is a function of the $\bar{x}_j$ variables and $A_j$ is a function of the $x_j$ variables. Calculate $\dfrac{\partial \bar{A}_i}{\partial \bar{x}_m}$.