SUPPLEMENTARY TO LECTURE NOTES

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I. SUPPLEMENTARY TO THE LECTURE 1

See Figs. 4, 8 and 9 mentioned in the text of Lecture 1:

Fig. 4. Redshift in spectra of remote objects (including quasars)

Fig. 8. Black body spectrum of CMB

Fig. 9. CMB maps

A. What is Cosmology?

I A 1 *"Astronomical time machine"*,
I A 2 *The Universe as a whole*,
I A 3 *What is known about the Universe?*,
I A 4 *The Big Bang model*,
I A 5 *Cosmological principle.*

1. "Astronomical time machine"

It is inevitable that an astronomer studies objects remote in time as well as in space. Light travels a distance of 300,000 kilometers in one second, or ten thousand billion kilometers in a year. The nearest star, Alpha Centauri, is 3 light years from us: we see it as it was three years ago. The nearest galaxy comparable to our own Milky Way, Andromeda, is at a distance of two million light years; we are seeing the Andromeda galaxy, a naked eye object in a dark sky, as it was when homo sapiens had not yet evolved. A large telescope is a time-machine that can take us part way to creation, to examine regions from which light emanated more than five billion years ago, before our sun had ever formed.

In cosmology we really can observe the remote past of our Universe!!!

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The Universe is very large, strange and very old four- dimensional object, containing enormous variety of objects. Enjoy looking, for example, at the galaxy Andromeda or the whole cluster of galaxies, where gravitational lenses magnify images of galaxies.
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2. The Universe as a whole

Cosmology studies the large scale properties of the Universe as a whole: the origin, evolution and ultimate fate of the entire Universe. In cosmology the theory of the origin and evolution of the Universe is confronted with observations. Depending on the outcome of the observations, the theories will need to be abandoned, revised or extended to accommodate the data. The Universe denotes everything that is or ever will be observable, so that we can never hope to study another universe.
The size scale of the observable now is about $L_{obs} \sim 10^4$ Mpc. In order of magnitude this size scale is the product of the age of the Universe and the velocity of light.

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According to the definition of 1 parsec (pc) $L = 1$ pc, if $\alpha = 1$ arcsec, where α is called parallax.
In cosmology we deal mostly with distances measured in Mpc's: Mpc $= 10^6$ pc $= 3 \times 10^{22}$ m.
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Uniqueness of the Universe.
Unlike other branches of science, cosmology deals with an unique object, since there is only one Universe available for study. We can never know how unique is our Universe, for we have no other universe with which to compare. [Nevertheless, recently some cosmologists started a discussion about Multiverse in contrast to the Universe.]

The laws of physics.
The laws of physics which are locally measured in the laboratory have more general applicability. Cosmology is developed by extrapolation of locally verified laws of physics to remote locations in space and time.

Simplicity in average.
In Cosmology, simplicity is sought on sufficiently large scales. This scale is about 100 million pc.

3. What is known about the Universe?

The Universe contains everything and at first sight it should be too complicated to be successfully studied in a one term course. Fortunately, as we will see later, the Universe as a whole turns out to be simple enough. What is known at the present moment about the Universe as a whole? We know that
(i) The Universe is expanding from a hot and dense initial state called the Big Bang.
(ii) In the Big Bang the light elements were synthesized.
(iii) We believe that there was a period of inflation which led to many observable properties of the Universe.
(iv) Any observable large scale structure was seeded by some small perturbations which are the relics of quantum fluctuations.
(v) This structure is dominated by cold dark matter.
(vi) At the present moment the Universe seems to be expanding with an acceleration rather than a deceleration as cosmologists used to think in the 20th century.

4. The Big Bang model

According to the Big Bang Model model, 14.7 billion years ago, the portion of the Universe we can observe at the present moment was, say, just a few millimeters across. The Universe has since expanded from this hot dense state into the vast and much cooler cosmos we currently inhabit. We can see remnants of this hot dense matter as the now very cold cosmic microwave background radiation which is also called "relic radiation". This radiation can be "seen" by microwave detectors as a uniform glow across the entire sky.

5. Cosmological principle

The simplicity of the Universe is based on the following cosmological principle: **The Universe, on average, looks the same from any point.** If the universe is locally isotropic, as viewed from any point, it also should be uniform. So the cosmological principle states: **Our Universe is approximately isotropic and homogeneous, as viewed by any observer at rest. More accurately, the matter in the Universe is homogeneous and isotropic when averaged over very large scales.** This assumption is being tested continuously as we actually observe the large scale distribution of galaxies.
In cosmology the simplicity of the Universe appears on sufficiently large scales. A few decades ago it was assumed that homogeneity applies on scales above 10 Mpc. However, the recent discovery of giant filaments and voids, as well as large-scale streaming motions, suggests that one may need to go to scales of order 100 Mpc. Since the scale of the observable Universe is around 6000 Mpc, this is only a factor of 60 larger, still it gives about $60^3 \approx 2 \times 10^5$ cells, which is large enough number to apply statistics.
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A long time ago cosmologists believed in a much stronger principle, called the perfect cosmological principle, which says that the Universe appears the same from all points and at all times. In other words, there can have been no evolution. This contradicts the observations which show that the real Universe does not satisfy this principle because we now have evidence that the Universe evolves (expansion of the Universe).
Another principle, the status of which remains ambiguous, is the so-called Anthropic Principle. This principle claims that certain features of the Universe - such as the values of the physical constants - are determined by the requirement that life should arise, because otherwise we could not be here asking questions about it.
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B. Cosmography of the Universe

1. Length scales in the Universe

At the current epoch the Universe exhibits many scales of structure and we will start off by specifying the typical length scales associated with these structures.

The typical separation between stars in the disk of the Milky Way (our own galaxy) is around a parsec. For comparison, the distance from the Earth to the Sun, which defines the Astronomical Unit, is 1 AU $= 1.5 \times 10^{11}$ m $=$ 0.000005 pc. Most of the stars in our Galaxy - including the Sun - are contained in a disc with a radius of about 10 kpc $= 10^4$ pc and thickness of 300 pc; these are the so-called "Population I" stars. There is also a nuclear bulge and a spheroidal distribution of "Population II" stars; many of the latter are contained in Globular Clusters, each of which has a radius of about 10 pc and contains around 10^6 stars.
These features are common to all spiral galaxies. The typical separation between large galaxies like our own is around 1 Mpc $= 10^6$ *pc*, but many galaxies are assembled into groups or clusters where the separation may be much smaller. For example, the Milky Way(which would look very similar to the galaxy Andromeda if one observed it from the Andromeda) is part of a Local Group and this also comprises the Large Magellanic Cloud (LMC) at a distance of 55 kpc, the Small Magellanic Cloud (SMC) at 67 kpc, M31 (Andromeda) at 710 kpc, M33 at 850 kpc, and several dozen dwarf galaxies. Other nearby groups are M81 at a distance of 2.9 Mpc and M101 at 6.8 Mpc. About 1% of galaxies are contained in much larger groups - containing hundreds or even thousands of members - and such "clusters" have scales of order a Mpc. For example, the Virgo cluster at a distance of about 20 Mpc contains hundreds of galaxies, while the Coma cluster at a distance of 100 Mpc contains thousands. It is now known that clusters themselves may clump into "superclusters" with scales of order 10 Mpc. In particular, our own Local Group is part of the Virgo Supercluster. We believe that Superclusters contain considerable amount of dark matter. The recent redshift surveys, which give 3-dimensional information about the Universe, indicate that there is structure on even larger scales than this . Most galaxies seem to lie on giant sheets or filaments with scales of up to 100 Mpc. There are hints of this even from the projected distribution of galaxies in the sky but one needs distance information - which can only come from measuring redshifts - to confirm the reality of this structure. Superclusters correspond to places where the sheets intersect and, in between the sheets, are giant voids. Some people claim that the voids are distributed like bubbles, with an average bubble radius of about 20 Mpc. Only on scales larger than about 100 Mpc is it safe to assume that the Universe is smooth and unstructured. At the present moment we use results of very deep surveys with measuring high red-shifts of remote galaxies. For comparison, the radius of the **observable** Universe (i.e. the distance light could have travelled since the Big Bang) will be shown to lie between 6000 and 10^4 Mpc. The Universe itself may be much larger than this and possibly infinite in extent.

2. Mass scales of the Large Scale Structure (LSS)

Let us now specify the mass scales associated with these scales of structure. The mass of the Sun 1 $M_\odot = 2 \times 10^{30}$ kg is a convenient unit to use for this purpose. Recall that all stars have a mass in the range 0.1 $M_\odot$ to 100 $M_\odot$, smaller objects being too small to ignite their nuclear fuel and larger ones being unstable to pulsations. Globular Clusters have a mass of around 10^6 $M_\odot$, while spiral galaxies like our own all have a mass of order 10^{11} $M_\odot$. There are other sorts of galaxies and these span a much larger mass range - from 10^8 $M_\odot$ to 10^{12} $M_\odot$, but spirals are most numerous. Clusters have a mass of around 10^{13} $M_\odot$, superclusters around 10^{15} $M_\odot$, and filaments around 10^{17} $M_\odot$. The visible Universe itself has a total mass of around 10^{22} $M_\odot$, so the number of galaxies in the Universe is comparable to the number of stars in the Galaxy.

C. Velocities in the Universe

1. Peculiar velocities

Each object in the Universe moves with some peculiar velocity associated with the different scales of structure. This is the name given to the motion of a galaxy due to its rotation and motion as influenced by the gravitational pull of nearby clusters. For example, the Earth's equatorial rotation speed is 0.3 km/s, the escape speed from the Earth is 10 km/s, the speed of the Earth around the Sun is 30 kms/s, the speed of the Sun around the centre of the Galaxy is 250 km/s, disc stars also have random motions (i.e. a velocity dispersion) of around 20 km/s superposed on this general rotation and in elliptical galaxies the stellar velocities are of order 100 km/s but most of this is in random motion since there is little rotation. Our Galaxy itself has a peculiar velocity of around 100 km/s relative to the Local Group; one can understand this as arising from the net gravitational pull of all the other members of the group. Within rich clusters galaxies may have velocity dispersions as high as 1000 km/s. The large-scale redshift surveys show that

clusters and superclusters themselves have peculiar motions. For example, the Local Group and possibly the entire Virgo supercluster has a peculiar velocity of 600 km/s relative to the cosmic microwave background radiation.
In any gravitationally bound system the characteristic velocity V associated with a system of size R, mass M and average density $\rho \sim M/R^3$ in order of magnitude is determined from

$$V \sim a_{grav} \times \tau, \text{ where the gravitational acceleration } a_{grav} \sim \frac{GM}{R^2}, \text{ and time scale } \tau \sim \frac{R}{V}. \tag{C.1}$$

$$\text{Thus } V \sim \frac{GM}{R^2} \cdot \frac{R}{V}, \text{ hence } V \sim \sqrt{\frac{GM}{R}} \sim R\sqrt{G\rho}. \tag{C.2}$$

These are only order-of-magnitude relations but they can be specified more precisely in any particular context. The quantities V and R can usually be measured observationally and a dark matter problem arises whenever the value of M or ρ exceeds the mass or density in visible form. If an object is not a member of a gravitationally bound system the previous arguments for estimating the peculiar velocity should be slightly modified: instead of time scale τ we should take the age of the Universe, t_0: mass M at distance R induces acceleration, a_{grav}, but during the whole life of the Universe the gain of velocity after time equal to the age of the Universe is

$$v = \frac{GM}{R^2} t_0. \tag{C.3}$$

This order of magnitude estimate neglects the expansion of the Universe (we assume that the distance R is nearly the same during time t_0), however this is a good estimate, since the expansion reduces v only by some numerical factor.

2. Hubble velocities

At the beginning of 20^{th} the idea that the universe was expanding was thought to be absurd. However, in 1929, Edwin Hubble announced that his observations of galaxies outside our own Milky Way showed that they were systematically moving away from us with a speed that was proportional to their distance from us. The more distant the galaxy, the faster it was receding from us. The universe was expanding! Hubble observed that the light from a given galaxy was shifted further toward the red end of the light spectrum the further that galaxy was from our galaxy. This is Doppler red-shift. The specific form of Hubble's expansion law is important: the speed of recession is proportional to distance. This suggests a homogeneous, isotropic, and expanding universe. Edwin Hubble was observing a group of objects known as spiral nebulae (spiral galaxies). These objects contain a very important class of stars known as Cepheid Variables. Because the Cepheids have a characteristic variation in brightness, Hubble could recognize these stars at great distances and then compare their observed luminosity to their known luminosity. This allowed him to compute the distance to the stars, since luminosity is inversely proportional to the square of the distance. The intrinsic, or absolute, luminosity is calculated from simple models that have been commensurate with observations of near Cepheids. When Hubble compared the distance of the Cepheids to their velocities (computed by the redshift of their spectrum) he found a simple linear relationship: $\vec{v} = H\vec{r}$, where $\vec{v}$ is the velocity of the galaxy, H is the so-called Hubble constant. It is clear that time scale of expansion, i.e. the age of the Universe is of order $t_0 \sim \frac{|\vec{r}|}{|\vec{v}|} = H^{-1}$. The Hubble constant is not actually a constant, but can be a function of time depending on the chosen model. The standard notation is to adopt H_0 as the 'current' observed Hubble parameter, whereas H = H(t) is referred to as the Hubble constant. At the beginning the value of the Hubble parameter, obtained from the original Hubble's observation was to large and the age of the Universe was too short (shorter than the age of the Earth). As we now understand, this was related to some mistakes in determination of distances. The current accepted value of the Hubble parameter is $H_0 = 100kms^{-1}Mpc^{-1}h_0$, where $h_0 \approx 0.7$. The motion of any object in the Universe is a superposition of peculiar velocity and the velocity which is related to the expansion of the Universe: $V_{tot} = H_0D + V_{pec}$. Peculiar velocities don't exceed 500 km s^{-1}, and can be neglected at far distances (since the Hubble velocity is proportional to the distance), if $H_0D >> 500$ kms^{-1}. The Hubble velocity will only dominate on scales exceeding $D \approx V_{pec}/H_0 \approx 20 \div 40$ Mpc.

D. Content and brief history of the Universe

1. Visible versus invisible in the Universe

We know that atoms, the building blocks of planets, stars, galaxies, clusters of galaxies contribute only 4% to the total density of the Unverse. Visible baryonic matter provides only less than 1% of the total density of the Universe. Dark matter comprises 22% of the Universe. This invisible matter does not emit or absorb light. It has only been detected indirectly by its gravity.i.e. due to gravitational effects on visible matter. Dark matter can be baryonic or non-baryonic (i.e. exotic, hypothetical). Dark baryonic matter provides another 3% of the total density ($4\% - 1\%$). Thus the bulk of dark matter is non baryonic. We know that 74% of the Universe, is composed of so called "dark energy" that acts as a sort of an anti-gravity. The dark energy is responsible for the present-day acceleration of the universal expansion. Content of the Universe varies in the course of its evolution.

2. Dark matter

What is the nature of the "dark matter"? There are a number of candidates for the dark matter: An example of ordinary dark matter is MACHOs (MAssive Compact Halo Objects). Halos surround galaxies. MACHOs can be detected by gravitational lensing experiments. MACHOs could be brown dwarfs, objects whose mass is not large enough and temperatures in their cores are not high enough to ignite nuclear reactions in their cores, so that these objects are practically invisible. The role of MACHOs could be played by hypothetical primordial black holes. An example of exotic dark matter is WIMPs (Weakly Interacting Massive Particles). These hypothetical particles interact so weakly with ordinary matter that the only way to detect them in astronomy is to search their gravitational effects on visible matter. There is also some chance to produce these particles in the laboratory (this one of the scientific objectives of "supercolliders").

3. Olber's paradox and expansion of the Universe

Expansion of the Universe can help to resolve what is termed Olber's paradox. This concerns the question of why the sky is dark at night. In an infinite static Universe with a uniform density of sources, the sources would cover the whole sky when R got too large, so one could expect that the surface brightness of the sky be the same as brightness of the Sun. This paradox is resolved in the Big Bang picture, partly because the Universe has a finite age (so that one never sees sources much more distant than the Hubble scale) and partly because the flux from remote sources is reduced by redshift due to expansion of the Universe.

4. Brief history of the Universe

The cosmological principle allows the evolution of the universe. In other words, according to cosmological principle the Universe is not the same at different moments of time. The universe has its history. The expansion of the universe over most of its history has been relatively gradual, however a rapid period, "inflation", preceded the Big Bang expansion. The CMB radiation was emitted only a few hundred thousand years after the Big Bang, long before stars or galaxies ever existed. Since the universe was very hot through most of its early history, there were no atoms in the early Universe, only free electrons and nuclei. The cosmic microwave background photons easily scatter off of electrons. Eventually, the universe cooled sufficiently that protons and electrons could combine to form neutral hydrogen. This was thought to occur roughly 400,000 years after the Big Bang. After that the Universe is absolutely transparent for CMB photons which practically don't interact with neutral hydrogen. Thus we can look through the universe back in time, what we actually see is called "the surface of last scattering".

E. Cosmic Microwave Background (CMB) radiation

1. Discovery of Cosmic Microwave Background (CMB)

The CMB radiation was discovered in 1965. Penzias and Wilson shared the 1978 Nobel prize in physics for this discovery. The CMB is the relic radiation left over from the hot Big Bang. Its temperature, $T = 2.725$ Kelvin, is extremely uniform all over the sky. The energy (or frequency) spectrum of this radiation corresponds to black-body radiation. At the present moment the error bars on the data points are so small that they can not be seen on the curve predicted by the Big Bang theory and there is no alternative theory yet proposed which predicts the same energy spectrum, hence the measurement of the CMB spectrum is the test of the Big Bang theory of expanding Universe. It is known that approximately 400,000 years after the Big Bang the temperature of the CMB dropped below $3000K$. Starting from this moment photons of the CMB had the average energy so small that they could not prevent protons and electrons from forming neutral hydrogen. Then the CMB photons interacted very weakly with neutral hydrogen. In other words, starting from this moment the Universe started to be opaque for the CMB photons. If we look at different directions over sky we can see back to this moment of time only. It seems that we see some sort of a surface which, as we know from the previous lecture, is called the surface of the last scattering since it was the last time most of the CMB photons scattered by free electrons. All maps of the temperature of the CMB can be considered as direct images of this surface of last scattering. [See WMAP site).]

2. Anisotropy of CMB

For a long time after its discovery the CMB radiation did seem to be uniform over sky. When observational technology improved, dipole anisotropy was detected. This anisotropy is related to the peculiar velocity of the Earth (the Sun), v with respect to the CMB, i.e. relative to the frame of reference in which the CMB has no dipole anisotropy: $\frac{\delta T_D}{T} \sim \frac{v}{c} \approx 10^{-3}$. Approximately 20 years later, in 1992, the Cosmic Background Explorer (COBE) satellite detected cosmological fluctuations in the microwave background temperature. Mathner and Smooth received the 2006 Nobel Prize in Physics (the second one for CMB) for their discovery of the black-body spectrum and the anisotropy of the CMB radiation using the Cosmic Background Explorer (COBE). Only very sensitive instruments, such as the COBE and even more sensitive the WMAP, can detect tiny fluctuations in the cosmic microwave background temperature. These fluctuations brings direct information about the physical conditions in the very early Universe, about the nucleosynthesis (i.e. origin of chemical elements) and the origin of galaxies and formation of the Large Scale Structure of the Universe, in order words, with the help of the CMB fluctuations one can determine the basic parameters of the Big Bang theory. It is possible to say that any map of the CMB temperature is a direct image of the remote past of our Universe (see lecture 1 about an astronomical time machine). These fluctuations, $\frac{\delta T}{T} \sim 10^{-5}$, contain extremely valuable information about the origin, evolution, and content of the universe. These cosmic microwave temperature fluctuations are believed to trace fluctuations in the density of matter in the early universe, as they were imprinted shortly after the Big Bang. The "angular spectrum" of the fluctuations (obtained by the expansion of temperature field over spherical harmonics) in the WMAP full-sky map shows the relative brightness of the "spots" in the map vs. the size or multipole numbers l in expansion of the temperature anisotropy over spherical functions. The shape of this curve contains important information about the history of the Universe. Thus the CMB has a very exiting history. But this is not the end of the story.

3. Polarization of CMB

A more detailed picture of the early Universe is obtained if we measure the polarization of the CMB. This polarization is unavoidably generated due to scattering of the CMB photons on free electrons during the epoch of highly ionized plasma. This polarization was measured first on the South Pole in 2002. Thus the first map of polarization was obtained. WMAP has produced maps of anisotropy-polarization correlations over the whole sky. The new information obtained with the help of polarization measurements and some theoretical models provides new clues about events which took place when the the age of the Universe was only tiny fraction of second. In the future we expect even the detection of cosmological gravitational waves, because they generate special type of polarization called B-mode polarization.

F. Examples, problems and summary

1. How to state the cosmological principle?

Problem:
(i) Explain briefly what is meant by the cosmological principle and (ii) what observational data support it. (iii) What is the approximate scale at which homogeneity is reached?
Solution:
(i) The cosmological principle says: The Universe is the same everywhere. (ii) The expansion of the Universe, the evolution of objects in the Universe and the isotropy of Cosmic Microwave Background support the cosmological principle. (iii) This scale is about 100 Mpc.

2. What is the distance between dark objects?

Problem: Assume that a small fraction of the matter density in the Universe can be explained by some hypothetical dark objects of mass $M = 10^{-3}M_{\odot}$, where $M_{\odot}$ is solar mass (these objects are as massive as the Jupiter). The contribution of these objects to the density of the Universe is $\rho_{obj} = 10^{-34}$ kg m^{-3}. Estimate the average distance between these objects.
Solution: The number density of these objects is

$$n_{obj} = \frac{\rho}{M_{obj}}, \tag{F.1}$$

and the average distance between these objects is determined from

$$d_{obj}^3 n_{obj} \approx 1. \tag{F.2}$$

From (F.2) we obtain

$$d_{obj} = n_{obj}^{-1/3} = \left(\frac{M_{obj}}{\rho}\right)^{1/3} = \left(\frac{10^{-3}M_{\odot}}{10^{-34}\ \mathrm{kg\ m^{-3}}}\right)^{1/3} = \left(\frac{10^{-3}\times 2\times 10^{30}\ \mathrm{kg}}{10^{-34}\ \mathrm{kg\ m^{-3}}}\right)^{1/3}. \tag{F.3}$$

Finally,

$$d_{obj} = \left(2\times 10^{61}\right)^{1/3}\ \mathrm{m} \approx 3\times 10^{20}\ \mathrm{m} \approx 10\ \mathrm{kpc}. \tag{F.4}$$

3. Can you write Hubble law in vector form?

Problem:(i) Write down the Hubble law in vector form. (ii) Consider three galaxies in an expanding Universe located at points a, b and c. Prove that if the Hubble law is valid for an observer at a, then it is also valid for observers at b and at c. (iii) Assume that the vector $\vec{r}_{ab}$ is perpendicular to the vector $\vec{r}_{ac}$. For an observer in the galaxy a the galaxy b is redshifted with $z^b_{(a)} = 0.6$ and the galaxy c is redshifted with $z^c_{(a)} = 0.8$. Find the redshift $z^c_{(b)}$ of the galaxy c, measured by an observer in the galaxy b.
Solution: (i) The Hubble law in vector form is

$$\vec{v} = H\vec{r}. \tag{F.5}$$

(ii) Using this vector form, we have for the observers in a and b

$$\vec{v}_{ab} = H\vec{r}_{ab}, \quad \vec{v}_{ac} = H\vec{r}_{ac}, \quad \vec{v}_{ba} = -\vec{v}_{ab} = -H\vec{r}_{ab} = H\vec{r}_{ba}, \tag{F.6}$$

$$\text{and}\quad \vec{v}_{bc} = \vec{v}_{ba} + \vec{v}_{ac} = -\vec{v}_{ab} + \vec{v}_{ac} = -H\vec{r}_{ab} + H\vec{r}_{ac} = H(-\vec{r}_{ab} + \vec{r}_{ac}) = H(\vec{r}_{ba} + \vec{r}_{ac}) = H\vec{r}_{bc}. \tag{F.7}$$

Then, for the observer c we have

$$\vec{v}_{ca} = -\vec{v}_{ac} = -H\vec{r}_{ac} = H\vec{r}_{ca} \quad \text{and} \quad \vec{v}_{cb} = -\vec{v}_{bc} = -H\vec{r}_{bc} = H\vec{r}_{cb}. \tag{F.8}$$

(iii) Redshif z is related with velocity as

$$z = \frac{v}{c} = \frac{Hr}{c}, \tag{F.9}$$

Hence

$$\left[z^c_{(b)}\right]^2 = \left(\frac{H}{c}\right)^2 [r_{cb}]^2 = \left(\frac{H}{c}\right)^2 \left[[r_{ca}]^2 + [r_{ab}]^2\right] = \left(\frac{H}{c}\right)^2 [r_{ac}]^2 + \left(\frac{H}{c}\right)^2 [r_{ab}]^2 = \left[z^c_{(a)}\right]^2 + \left[z^b_{(a)}\right]^2. \tag{F.10}$$

$$\text{Finally } z^c_{(b)} = \sqrt{(0.8)^2 + (0.6)^2} = \sqrt{0.64 + 0.36} = 1. \tag{F.11}$$

4. Why night sky is dark?

Problem: (i) Show that in an infinite static Universe with a uniform density of sources the night sky should be as bright as the Sun (the Olber's paradox). (ii) Explain qualitatively why the evolution and expansion of the Universe can resolve this paradox.
Solution: (i) In an infinite static Universe with a uniform density of sources n, the number of sources per steradian in a shell of radius r and thickness dr centered around an observer at O is

$$dN = nr^2 dr. \tag{F.12}$$

The total intensity from all the sources with luminosity P within radius R is

$$I = n \int_0^R r^2 P/r^2 dr = nPR. \tag{F.13}$$

Formally this diverges as $R \to \infty$. More accurately, sources would cover the sky when $R \sim (nr_s^2)^{-1}$ and one would expect the night sky should be as bright as the Sun. This is the paradox. (ii) The Olber's paradox can be resolved partly due to evolution, since the Universe has a finite age and there are no sources more distant than the Hubble scale, partly due to expansion of the Universe, since the flux from the sources is reduced by redshift.

5. Useful questions to think about on your own

Questions:
(i) The ionization energy of hydrogen is $E = 13.6$eV. What is the corresponding temperature?
(ii) Explain qualitatively why hydrogen is fully ionized even when the temperature is considerably smaller than E.
(iii) Why is the epoch of recombination also called the decoupling epoch?
(iv) Explain how the anisotropy of the microwave background yields information about the physical conditions in the universe at the moment of decoupling.

6. Summary

(i) The Universe is homogeneous (cosmological principle) at scales about 100 Mpc.
(ii) The Universe is expanding according to Hubble law.
(iii) The Universe at the present epoch is expanding with acceleration, which means that dynamically it is dominated by dark energy.
(iv) The structure of Universe is dominated by cold dark matter.
(v) The number density of CMB photons is about 10^9 larger than the number density of barions.
(vi) CMB is the main source of information about the past of the Universe.

II. SUPPLEMENTARY TO THE LECTURE 2

03 October 2011: Nobel physics prize honours accelerating Universe finding, i.e the discovery of dark energy!!!

"The two groups announced their results within just weeks of each other and they agreed so closely; that's one of the things that made it possible for the scientific community to accept the result so quickly."

A. Homogeneity and Hubble Law

1. Lagrangian coordinates versus Eulerian coordinates

The Eulerian coordinates is a way of looking at fluid motion that focuses on specific locations in the space through which the fluid flows. [This can be visualized by sitting on the bank of a river and watching the water pass the fixed location.]
The Lagrangian coordinates (or the co-moving coordinates) is a way of looking at fluid motion where the observer follows individual fluid particles as they move through space and time. Plotting the position of an individual particle through time gives the path-line of the particle.[This can be visualized by sitting in a boat and drifting down a river.] The Lagrangian coordinates are like "labels" for all objects, particles in the Universe or elements of fluid, i.e. when given to some object once this label is associated with this object for ever (like numbers on shirts of football players). In other words, Lagrangian coordinates are constants for any selected object. In other words, the fundamental difference between Lagrangian and Eulerian coordinates is that for an arbitrary moving object the Lagrangian coordinates are constants, while the Euilerian coordinates could be arbitrary and rather complicated functions of time.

2. The case of spherically symmetric motion

Let us choose spherical coordinates with the center at the point of our location. The position of any physical object or any point in the Universe can be characterized by some radius-vector corresponding to Eulerian coordinates, $\vec{r}$. In terms of Lagrangian coordinates, which we denote as $\vec{\chi}$, the radius-vector of any selected element is

$$\vec{r} = \vec{r}(t, \vec{\chi}) \equiv \vec{r}(t, \chi_1, \chi_2, \chi_3). \tag{A.1}$$

Taking into account that all the motions we consider in this section are radial, we can say that

$$\vec{r}(t, \vec{\chi}) = \Phi(t, \vec{\chi})\vec{r}(t_0, \vec{\chi}), \tag{A.2}$$

where t_0 is an arbitrarily chosen initial moment of time and $\Phi(t, \vec{\chi})$ is a scalar function. Then the velocity of selected element is

$$\vec{v}(t, \vec{\chi}) = \frac{\partial \Phi(t, \vec{\chi})}{\partial t}\vec{r}(t_0, \vec{\chi}) \equiv \dot{\Phi}(t, \vec{\chi})\vec{r}(t_0, \vec{\chi}). \tag{A.3}$$

3. Hubble law and scale factor

If we put Eqs (A.2) and (A.3) into the Hubble law in the vector form,

$$\vec{v} = H\vec{r}, \text{ hence } \dot{\Phi}(t, \vec{\chi})\vec{r}(t_0, \vec{\chi}) = H(t)\Phi(t, \vec{\chi})\vec{r}(t_0, \vec{\chi}), \tag{A.4}$$

which after obvious contraction is reduced to a scalar equation

$$\dot{\Phi}(t, \vec{\chi}) = H(t)\Phi(t, \vec{\chi}). \tag{A.5}$$

Now we can obtain a general solution of this equation, indeed

$$\frac{\dot{\Phi}(t, \vec{\chi})}{\Phi(t, \vec{\chi})} = H(t), \text{ hence } [\ln \Phi(t, \vec{\chi})]^{\cdot} = H(t) \text{ and } \ln \Phi(t, \vec{\chi}) = \int_{t_*}^{t} H(t')dt' + \Psi(\vec{\chi}), \tag{A.6}$$

where t_* is an arbitrary moment of time and $\Psi(\vec{\chi})$ is an arbitrary function of Lagrangian coordinates $\vec{\chi}$ only, i.e. Ψ does not depend on time. As follows from Eq. (A.6)

$$\Phi(t,\vec{\chi}) = \exp\left[\int_{t_*}^{t} H(t')dt' + \Psi(\vec{\chi})\right] = a(t)f(\vec{\chi}), \tag{A.7}$$

where

$$f(\vec{\chi}) = \exp\left[\Psi(\vec{\chi})\right], \tag{A.8}$$

is also arbitrary function of $\vec{\chi}$ only, while

$$a(t) = \exp\left[\int_{t_*}^{t} H(t')dt'\right] \tag{A.9}$$

is a function of time only and does not depend on Lagrangian coordinates. Substituting Eq.(A.7) into Eq. (A.5), we obtain

$$\frac{\dot{a}(t)}{a(t)} = H(t). \tag{A.10}$$

Then substituting Eq.(A.7) into Eq. (A.2) we obtain

$$\vec{r}(t,\vec{\chi}) = a(t)\vec{q}(\vec{\chi}), \quad \text{where} \quad \vec{q}(\vec{\chi}) = f(\vec{\chi})\vec{r}(t_0,\vec{\chi}). \tag{A.11}$$

Eq.(A.11)can be considered as a transformation from one set of Lagrangian coordinates $\vec{\chi} \equiv (\chi_1, \chi_2, \chi_3)$ to another set of Lagrangian coordinates $\vec{q} \equiv (q_1, q_2, q_3)$. Hence, we can forget about $\vec{\chi}$ and express $\vec{r}$ in terms of $\vec{q}$ as

$$\vec{r}(t,\vec{q}) = a(t)\vec{r}(t_0,\vec{q}). \tag{A.12}$$

Eq. [A.10] can be considered as the definition of $a(t)$ in terms of the Hubble constant $H(t)$. And vice verse, if $a(t)$ is known we can calculate $H(t)$ using this equation. As follows from Eq. [A.12] it is reasonable to call $a(t)$ as the scale factor which describes the Universe as a whole because it does not depend on Lagrangian coordinates.

4. What is a cosmological model?

Let us assume that we labeled all elements of the Universe by Lagrangian coordinates q and know the scale factor, $a(t)$, at some moment of time, t_0. As follows from Eq.(A.12), this means that me know Euler positions of all these elements at this moment of time as well. Hence, if we know the dependence of function $a(t)$ on time we can reconstruct positions of all elements of the Universe in the past and predict these positions in the future. Then using the Hubble law we can obtain all Hubble velocities at any moment of time. It is obvious that any peculiar velocities can not be obtained in such a way, because in order to calculate them we need more information about the structure of the Universe and the physical properties of different objects. But at this stage we don't need to care about the peculiar velocities at all, because the averaging over scales of order 100 Mpc gives zero average for the peculiar velocities. When we say that H is called Hubble's constant, we mean that it is the same everywhere in the Universe and depends on time only. This is the homogeneity of the Universe, which obviously implies that all length scales change in the same way. Homogeneity also requires that the fluid density is the same everywhere at a given time and depends only on time. This statement actually is a definition of cosmic time: one can measure density to determine time. Thus we conclude that in order to describe the Universe as whole it is enough to determine the scale factor as a function of time. To do this we should write a complete set of mathematical equations which enable us to determine $a(t)$. Such a set of equations is called a cosmological model. The required equations come from physics. Hence the set of equations we use for construction of the cosmological model depends on the physical theory we use and assumptions we made within the framework of the chosen physical theory. In the next lecture we will work with Newtonian theory of gravity to obtain the Newtonian cosmological model.

B. Acceleration equation and conservation of mass

II B 1 *Newtonian gravity and homogeneous sphere of dust*,
II B 2 *Conservation of mass* ,
II B 3 *The final set of equations for Newtonian cosmological model.*

1. Newtonian gravity and homogeneous sphere of dust

The fact that the Universe is isotropic and homogeneous (the Cosmological Principle) leads to a very simple set of cosmological models. In this part we derive these models in the context of Newtonian theory starting not from energy conservation but from the second law of Newton for the motion of a small mass in spherically symmetrical gravitational field.
Let us now consider the equation of motion for a particle of mass Δm on the surface of the sphere with radius r and mass M. It is easy to prove that the gravitational field of a uniform medium external to a spherical cavity is zero. According to the Newtonian theory of gravity the gravitational field generated by any spherically symmetric distribution of mass within radius r is the same as if all mass was concentrated at the center. The second law of Newton combined with Newton's gravity law therefore implies that the equation of motion for the particle is

$$\Delta m\ddot{\vec{r}} = -\frac{GM\Delta m\vec{r}}{r^3}, \text{ and for the radial acceleration we obtain } \vec{a} \equiv \ddot{\vec{r}} = -\frac{GM\vec{r}}{r^3}. \tag{B.1}$$

We can see that Δm canceled. This is related to the fact that inertial and gravitational masses of any body are equal to each other. Homogeneity of the Universe requires that the fluid density is the same everywhere at a given time, so that

$$\rho(\vec{r}, t) = \rho(t), \text{ hence the mass is } M = \frac{4\pi\rho r^3}{3}. \tag{B.2}$$

Substituting Eq.(B.2) into Eq. (B.1) we obtain

$$3\vec{a} = -4\pi G\rho(t)\vec{r}. \tag{B.3}$$

Now we can use the expression for $\vec{r}$ in terms of time, t, and Lagrangian coordinates, $\vec{q}$ to obtain

$$\vec{r}(t, q) = a(t)\vec{q} \text{ and } \vec{a} \equiv \ddot{\vec{r}} = \ddot{a}(t)\vec{q}. \tag{B.4}$$

Finally, substituting Eq. (B.4) into Eq. (B.3) and taking into account that the Lagrangian vector $\vec{q}$ cancels, we obtain the following equation for the scale factor

$$3\ddot{a}(t) = -4\pi G\rho(t)a(t). \tag{B.5}$$

This is the so called acceleration equation. You can see that this equation is different from what was obtained in Lecture 2 by differentiating the Friedmann equation and taking into account the first law of thermodynamics. Thus, the Newtonian approach is not self-consistent and can not explain in this form the expansion with acceleration, because $\rho > 0$ corresponds $\ddot{a} < 0$.However, it gives correct results for the evolution of the Universe when the pressure of matter is negligible. In cosmology such kind of matter is called **dust**. As follows from the Hubble law $\vec{v} = H\vec{r}$ and the definition of scale factor $\dot{a}/a = H$, all Hubble velocities,

$$\vec{v}(t, \vec{q}) = H(t)\vec{r}(t, \vec{q}) = \frac{\dot{a}}{a} \cdot a\vec{q}, \tag{B.6}$$

are proportional to $\dot{a}$. Since $\ddot{a} < 0$ the derivative of the scale factor, $\dot{a}$, decays with time, which means that all Hubble velocities decrease in the course of expansion of the Universe. That is why we say that the Universe, according to the Newtonian theory prediction, should always expand with deceleration. The acceleration equation contains two unknown functions, $a(t)$ and $\rho(t)$, and can not be solved unless we supplement it with another equation relating $a(t)$ and $\rho(t)$. This supplementary equation can be obtained from the mass conservation law.

2. Conservation of mass

We assume that the work done by forces of pressure is negligible, otherwise the change in the internal energy of the fluid would also contribute to the change in M. This assumption is correct if all "particles" are non-relativistic, so that their total energy is dominated by their rest-mass energy. This means that mass is conserved, i.e.

$$M = \frac{4\pi\rho(t)r^3}{3} = \text{constant}, \quad \text{hence} \;\; \rho(t) = \rho(t_0)\frac{r^3(t_0,\vec{q})}{r^3(t,\vec{q})} = \rho_0\frac{a^3(t_0)q^3}{a^3(t)q^3} = \rho(t_0)\frac{a^3(t_0)}{a^3(t)}. \tag{B.7}$$

This is the conservation of mass equation. In the above equations the index "$_0$" corresponds to some arbitrary moment of time t_0, say to the present moment or, in other words, to the moment of observations.

3. The final set of equations for Newtonian cosmological model

Now we have the following set of three differential equations

$$H(t) = \frac{\dot{a}(t)}{a(t)}, \;\; \ddot{a}(t) = -\frac{4\pi G}{3}\rho(t)a(t) \;\; \text{and} \quad \rho(t) = \frac{A}{a^3(t)}, \quad \text{where} \;\; A = \rho(t_0)a^3(t_0). \tag{B.8}$$

These three equations contain three unknown functions, a, H and ρ and can be solved. We call these equations a cosmological model because a, H and ρ are the same everywhere and are fundamental quantities describing expansion of the Universe as a whole. From (B.8) we obtain the following equation for $a(t)$

$$\ddot{a} = -\frac{4\pi G\rho(t_0)a^3(t_0)}{3a^2}. \tag{B.9}$$

C. Friedman equation (FE) and cosmological parameters

II C 1 *Derivation of FE from the acceleration equation*,
II C 2 *Re-scaling of the scale factor*,
II C 3 *The most fundamental cosmological parameters* ,
II C 4 *FE and relationships between cosmological parameters.*

1. Derivation of FE from the acceleration equation

The acceleration equation is the second order differential equation. To integrate this equation one can multiply both sides of this equation by $\dot{a}$. Taking into account that

$$\ddot{a}\dot{a} = \frac{1}{2}\frac{d\dot{a}^2}{dt}, \quad \text{i.e.} \quad -\frac{\dot{a}}{a^2} = \frac{da^{-1}}{dt}, \tag{C.1}$$

We obtain

$$\frac{d}{dt}\left(\dot{a}^2 - \frac{8\pi GA}{3a}\right) = 0, \;\; \text{hence,} \;\; \frac{1}{2}\dot{a}^2 = \frac{4\pi G}{3}Aa^{-1} + C = \frac{4\pi G\rho a^2}{3} - \frac{1}{2}kc^2, \tag{C.2}$$

where C is an integration constant which is expressed for convenience as $C = -kc^2/2$. The first two terms look like the kinetic and potential energies per unit mass, this equation from mathematical point of view equivalent to the Newtonian energy equation with the integration constant representing the total energy per unit mass. However, a is the scale factor rather than the radius of any particular sphere. Even dimensions of a are not specified by the above equation.

2. Re-scaling of the scale factor

In this supplementary material we choose a to have the dimensions of length, in this case k is dimensionless and one can then produce re-scaling of scale factor such that k is 0, 1 or -1. We can write down the Friedman equation as

$$\dot{a}^2 = \frac{8\pi G\rho a^2}{3} - kc^2. \tag{C.3}$$

Let me remind you that we have the freedom in our choice of Lagrangian coordinates

$$r(t,\vec{q}) = a(t)q, \tag{C.4}$$

this means that we can re-scale the Lagrangian coordinate and the scale factor in a self-consistent way:

$$q = \alpha^{-1}\tilde{q}, \;\; a = \alpha\tilde{a}, \tag{C.5}$$

where α is an arbitrary constant. As a result the Eulerian coordinate is unchanged. The Friedman equation can be re-written in terms of the new scale factor as

$$\frac{\alpha^2\dot{\tilde{a}}^2}{\alpha^2\tilde{a}^2} = \frac{8\pi G\rho}{3} - \frac{kc^2}{\alpha^2\tilde{a}^2}, \;\; \text{or} \;\; \frac{\dot{\tilde{a}}^2}{\tilde{a}^2} = \frac{8\pi G\rho}{3} - \frac{kc^2}{\alpha^2\tilde{a}^2}. \tag{C.6}$$

We can define α as follows: $\alpha = \sqrt{k}$, if $k > 0$, α is arbitrary, if $k = 0$ and $\alpha = \sqrt{-k}$, if $k < 0$. Then we define a new constant of integration

$$\tilde{k} = \frac{k}{\alpha^2}, \tag{C.7}$$

which is equal to 1, if $k > 0$m, 0, if $k = 0$, and -1, if $k < 0$. Then we replace back $\tilde{k}$ by k and $\tilde{a}$ by a.

3. The most fundamental cosmological parameters

(i) **Hubble parameter,** H_0, which is defined as the value of Hubble constant at the present moment of time (the time of observations).
(ii) **Density parameter,** Ω_0**,** defined as a ratio

$$\Omega_0 = \frac{\rho_0}{\rho_{cr}}, \tag{C.8}$$

where $\rho = \rho(t_0)$ is the actual average density of the Universe observed at the present moment and

$$\rho_{cr} = \frac{3H_0^2}{8\pi G} \tag{C.9}$$

has dimensions of density and is determined by the value of the Hubble parameter.
Hence both these parameters can be determined directly from observations. We introduce also two other parameters which, as we will see in the next section, also can be determined from observations.
(iii) **Deceleration parameter q** defined as

$$q = -\frac{\ddot{a}a}{\dot{a}^2}. \tag{C.10}$$

We can see that q is a dimensionless parameter.
(iv) **k-parameter** appeared in the Friedman equation and, as we know from the previous section, can be equal to 0, -1 or $+1$.
There are several other cosmological parameters which correspond predominantly to the content of the Universe.

4. FE and relationships between cosmological parameters

In Cosmology the Friedman equation plays two roles:
1) It gives the law of expansion, $a(t)$ (see the next Lecture.)
2) Written down for the present moment, $t = t_0$, it gives the relationship between the fundamental cosmological parameters:

$$H_0^2 = \frac{8\pi G\rho_0}{3} - \frac{kc^2}{a_0^2}, \text{ hence, } H_0^2\left(\frac{8\pi G\rho_0}{3H_0^2} - 1\right) = \frac{kc^2}{a_0^2}, \text{ and finally } k = \left(\frac{H_0 a_0}{c}\right)^2 (\Omega_0 - 1)\,. \tag{C.11}$$

The relationship between the deceleration parameter q and the density parameter Ω_0 can be obtained from the acceleration equation:

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = \frac{4\pi G\rho_0 a\cdot a}{3\dot{a}^2} = \frac{4\pi G\rho_0 a^2}{3\dot{a}^2} = \frac{4\pi G\rho_0}{3H_0^2} = \frac{1}{2}\frac{\rho_0}{\rho_{cr}} = \frac{\Omega_0}{2}. \tag{C.12}$$

Due to these relationships q and k can also be determined from direct astronomical observations!

D. Solution of the FE and the fate of the Universe

II D 1 *Asymptotic behavior of the solution of FE,*
II D 2 *Parametric solution of the FE with $k \neq 0$,*
II D 3 *Three types of Newtonian cosmological models.*

1. Asymptotic behavior of the solution of FE

Let us write down the Friedman equation in the following form

$$\dot{a}^2 = c^2\left(\frac{a_*}{a} - k\right), \text{ where } a_* = \frac{8\pi G\rho_0 a_0^3}{3c^2}. \tag{D.1}$$

To solve the Friedman equation let us use the method of separation of the variables:

$$\left(\frac{da}{dt}\right)^2 = c^2\left(\frac{a_*}{a} - k\right), \quad \frac{da}{dt} = \pm c\sqrt{\frac{a_*}{a} - k}. \tag{D.2}$$

We should take "+" here because we deal with an expanding rather than a contracting Universe.

$$dt = +\frac{da}{c\sqrt{\frac{a_*}{a} - k}} \text{ and } t = \frac{1}{c}\int_0^a \frac{da'}{\sqrt{\frac{a_*}{a'} - k}}. \tag{D.3}$$

Let us consider the asymptotic behavior of the Friedman equation and its solution for small a, i.e in the beginning of the expansion of the Universe when

$$\frac{a_*}{a} \gg |k|, \text{ i.e. } a \ll \frac{a_*}{|k|}. \tag{D.4}$$

If $k = 0$ this inequality is always valid. If $k \neq 0$ asymptotically the Friedman equation is the same as in the case $k = 0$. This asymptotic solution can be easily found, indeed:

$$t = \frac{1}{c}\int_0^a \frac{da'}{\sqrt{\frac{a_*}{a'}}} = \frac{1}{ca_*^{1/2}}\int_0^a a'^{1/2}da' = \frac{2a^{3/2}}{3ca_*^{1/2}}, \tag{D.5}$$

hence the asymptotic solution goes as

$$a \sim t^{2/3}, \text{ if } a \ll \frac{a_*}{|k|}. \tag{D.6}$$

2. Parametric solution of the FE with $k \neq 0$

The following relationships between between hyperbolic and trigonometric functions help a lot with solving the Friedman equation in the case when $k \neq 0$:

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}, \quad \cos x = \frac{e^{ix} + e^{-ix}}{2} \tag{D.7}$$

$$\sinh x = \frac{e^{x} - e^{-x}}{2} \quad \cosh x = \frac{e^{x} + e^{-x}}{2} \tag{D.8}$$

hence

$$\sin(ix) = \frac{e^{i(ix)} - e^{-i(ix)}}{2i} = \frac{e^{-x} - e^{x}}{2i} = -\frac{e^{x} - e^{-x}}{2i} = -\frac{1}{i}\sinh x = i\sinh x \tag{D.9}$$

$$\cos(ix) = \frac{e^{i(ix)} + e^{-i(ix)}}{2} = \frac{e^{-x} + e^{x}}{2} = \frac{e^{x} + e^{-x}}{2} = \cosh x \tag{D.10}$$

then

$$\cosh^2 x - \sinh^2 x = \cos^2(ix) - \left(\frac{\sin(ix)}{i}\right)^2 = \cos^2(ix) + \sin^2(ix) = 1. \tag{D.11}$$

Returning back to Eq. (D.3), we can use the following substitution:

$$a(\eta) = \frac{a_*}{k}\sin^2\frac{\sqrt{k}\eta}{2} = \frac{a_*}{2k}(1 - \cos\sqrt{k}\eta), \tag{D.12}$$

where η is a new variable. We should not worry that the argument $x = \sqrt{k}\eta$ could be a complex number because all sines and cosines can be expressed in terms of the hyperbolic functions of a real argument. Then we have

$$t = \frac{a_*\sqrt{k}}{2kc}\int_0^\eta \frac{\sin\sqrt{k}\eta' d\eta'}{\sqrt{\frac{a_* k}{a_* \sin^2\frac{\sqrt{k}\eta'}{2}} - k}} = \frac{a_*}{2kc}\int_0^\eta d\eta' F(\sqrt{k}\eta'), \tag{D.13}$$

where

$$F(x) = \frac{\sin x}{\sqrt{\frac{1}{\sin^2\frac{x}{2}} - 1}} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}\sin\frac{x}{2}}{\sqrt{1 - \sin^2\frac{x}{2}}} = 2\sin^2\frac{x}{2} = 1 - \cos x. \tag{D.14}$$

Hence

$$t = \frac{a_*}{2kc}\int_0^\eta d\eta'(1 - \cos\sqrt{k}\eta') = \frac{a_*}{2kc}(\eta - \frac{\sin\sqrt{k}\eta}{\sqrt{k}}). \tag{D.15}$$

Finally we have obtained the solution of Friedman equation in the following parametric form:

$$a = \frac{a_*}{2k}(1 - \cos\sqrt{k}\eta), \tag{D.16}$$

$$t = \frac{a_*}{2kc}(\eta - \frac{\sin\sqrt{k}\eta}{\sqrt{k}}). \tag{D.17}$$

3. Three types of Newtonian cosmological models

Thus according to Newtonian theory, there are three types of cosmological models:

$$\begin{array}{ll} 1) & k=0 \quad \Omega_0=1, \\ 2) & k=-1 \quad \Omega_0<1, \\ 3) & k=1 \quad \Omega_0>1,. \end{array} \tag{D.18}$$

Let us consider all these models separately.
(i) $k=0$**.** In this case the asymptotic solution obtained in section 10.1 is always valid and we have explicit solution

$$t=\frac{2a^{3/2}}{3ca_*^{1/2}}, \quad a=a_*\left(\frac{t}{t_*}\right)^{2/3}, \quad \text{where } t_*=\frac{2a_*}{3c}. \tag{D.19}$$

From the mass conservation equation (see the previous Lecture) using Eqs. (D.2) and (D.25), we obtain the following expression for ρ:

$$\rho=\rho_0\left(\frac{a_0}{a}\right)^3=\rho_0\left(\frac{a_0}{a_*}\right)^3\left(\frac{t}{t_*}\right)^{-2}=\frac{4}{9c^2}\rho_0 a_0^3(a_*)^{-1}t^2=\frac{1}{6\pi G t^2}. \tag{D.20}$$

(ii) $k=-1$**.** From the parametric solution obtained in the previous section we have

$$a=\frac{a_*}{-2}(1-\cos i\eta), \quad t=\frac{a_*}{-2c}(\eta-\frac{\sin i\eta}{i}), \tag{D.21}$$

From Eqs (D.14) and (D.15) we obtain

$$a=\frac{a_*}{2}(\cosh\eta-1), \quad t=\frac{a_*}{2c}(\sinh\eta-\eta). \tag{D.22}$$

If $a\to\infty$ the parameter η also goes to infinity and

$$a\approx\frac{a_*}{2}\frac{e^\eta}{2}, \quad t\approx\frac{a_*}{2c}\frac{e^\eta}{2}, \tag{D.23}$$

Thus if $a\gg a_*$ we have another explicit asymptotic solution

$$a=ct, \tag{D.24}$$

which corresponds to the asymptotically free expansion of the Universe, i.e. with an asymptotically vanishing deceleration. In other words the model with $k=-1$ expands in the future for ever, asymptotically approaching the so called Milne solution describing the expansion of an empty Universe.
(iii) $k=1$**.** In the case $k=1$ the performance of the solution is drastically different:

$$a=\frac{a_*}{2}(1-\cos\eta), \quad t=\frac{a_*}{2c}(\eta-\sin\eta). \tag{D.25}$$

We can see that a attains maximum at $\eta=\pi$ and then the Universe will start to contract until $a=0$. This is called the Big Crunch.
The behavior of all three cosmological models is predicted by the Newtonian theory.

E. General Relativity (GR) is required

II E 1 *Disagreement of Newton's theory with observations*,
II E 2 *Theoretical problems*,
II E 3 *Gravitational paradox in Newtonian theory*,
II E 3 *Expansion of the Universe with acceleration.*

Newton's law has since been superseded by Einstein's theory of general relativity, but it continues to be used as an excellent approximation of the effects of gravity. Relativity is only required when there is a need for extreme precision, or when dealing with gravitation for very massive objects. The Universe is really very very massive object.

1. Disagreement of Newton's theory with observations

(i) The predicted deflection of light by gravity using Newton's theory is only half the deflection actually observed.
(ii) Newton's theory does not fully explain the precession of the perihelion of the orbits of the planets, especially of planet Mercury. There is a 43 arcsecond per century discrepancy between the Newtonian prediction and the observed precession.
(iii) The observed fact that gravitational and inertial masses are the same for all bodies is unexplained within Newton's system and treated within this theory just as coincidence.

2. Theoretical problems

(i) There is no immediate prospect of identifying the mediator of gravity. Attempts by physicists to identify the relationship between the gravitational force and other known fundamental forces are not yet resolved.
(ii) Newton's theory requires that gravitational force is transmitted instantaneously. Newton himself felt the inexplicable action at a distance to be unsatisfactory. He was deeply uncomfortable with the notion of "action at a distance" which his equations implied. In 1692 he wrote:
"That one body may act upon another at a distance through a vacuum without the mediation of anything else, by and through which their action and force may be conveyed from one another, is to me so great an absurdity that, I believe, no man who has in philosophic matters a competent faculty of thinking could ever fall into it."

3. Gravitational paradox in Newtonian theory

The main problem in application of Newton' theory to cosmology is related with the fact that the Universe can be infinite. When we consider gravitational forces generated by infinite Universe the result depends crucially on the way how we integrate gravitational forces from different mass elements of the Universe. To illustrate this statement let us consider the so called shell theorem which says:
(i) A spherically symmetric body affects external objects gravitationally as though all of its mass were concentrated at a point at its center.
(ii) If the body is a spherically symmetric shell (i.e. a hollow ball), no gravitational force is exerted by the shell on any object inside, regardless of the object's location within the shell.
(iii) Inside a solid sphere of constant density the gravitational force varies linearly with distance from the center, becoming zero at the center of mass.

$$F = \frac{GMm}{r^2} = \frac{4\pi G\rho m r^3}{3r^2} \propto r. \tag{E.1}$$

If we integrate over spheres then contribution of outside spheres is exactly equal to zero and only finite part of the Universe contributes to gravitational force. In this case the force is finite but arbitrary. If we divide the Universe into infinite planes we will conclude that all bodies in the Universe should experience infinite disruptive gravitational forces. This is in the direct contradiction with the fact of our existence. Even the direction of this disruption depends on the way of integration. Not going into detail of relevant calculations we can conclude that Newtonian theory when applied to the infinite Universe is not self-consistent. This inconsistency is called "Gravitational paradox" or the second Olber's paradox.

4. Expansion of the Universe with acceleration

According to Newton's theory only matter density contributes into gravitational forces and the Universe should expand with deceleration. However, there is strong observational evidence that at present time our Universe is expanding with acceleration which implies some sort of repulsive gravitational forces. The substance generating such repulsive forces (anti-gravity) is the so called "dark energy". According to General relativity gravity is determined not only by density but by pressure and tension (which correspond to a negative pressure).

Tensions contribute to gravitational acceleration with opposite sign in comparison with density. This is the reason why according to General relativity our Universe can expand with acceleration.
This is very brief motivation why we should study General Relativity (GR) when we are trying to understand the structure and evolution of the Universe. According to GR gravitation is an attribute of curved spacetime instead of being due to a force propagated between bodies. In Einstein's theory, masses distort spacetime in their vicinity, and other particles move in trajectories determined by the geometry of spacetime. This allowed a description of the motions of light and mass that was consistent with all available observations.

F. Examples, problems and summary

II F 1 *Inflation*,
II F 2 *Conservation of mass*,
II F 3 *The Hubble constant as a function of the scale factor*
II F 4 *The parametric solution of the FE*,
II F 5 *Gravitational paradox*,
II F 6 *Summary.*

1. Inflation

Problem: Given that the Hubble constant, H, does not depend on time. Find the scale factor $a(t)$ as a function of time. Show that in this case the Universe expands with an acceleration.
Solution: From

$$\frac{\dot{a}}{a} = H, \;\; \dot{a} = Ha \;\text{ we have }\; a = a_0 e^{H(t-t_0)}. \tag{F.1}$$

Then we can see that in this case the Universe expands with acceleration:

$$\ddot{a} = H\dot{a} = H^2 a > 0. \tag{F.2}$$

2. Conservation of mass

Problem: Under what condition mass is conserved.
Solution: When pressure is negligible. Otherwise the forces of pressure produce a work, which change the energy within given volume. Energy E is related to mass M as

$$E = Mc^2, \tag{F.3}$$

Hence, the mass varies proportionally to the energy.

3. The Hubble constant as a function of the scale factor

Problem: Use FE for the Universe with $k = 0$ to find the Hubble constant H as a function of scale factor a if the present Universe contains only dust.
Solution: Taking into account that

$$H = \frac{\dot{a}}{a}, \;\; \rho_{cr} = \frac{3H_0^2}{8\pi G}, \;\text{ and }\; \rho = \frac{3H_0^2}{8\pi G}\left(\frac{a_0}{a}\right)^3, \tag{F.4}$$

FE can be written as

$$H^2 = \frac{8\pi G}{3} \cdot \frac{3H_0^2}{8\pi G}\left(\frac{a_0}{a}\right)^3, \;\text{ hence }\; H^2 = H_0^2\left(\frac{a_0}{a}\right)^3. \tag{F.5}$$

4. The parametric solution of the FE

Problem: Assume that the Universe is closed ($k = 1$) and contains only dust. Using FE and the energy conservation equation, verify that the evolution of the scale factor has the following parametric form:

$$a(\eta) = \frac{\beta}{2}(1 - \cos\eta), \quad t(\eta) = \frac{\beta}{2c}(\eta - \sin\eta). \tag{F.6}$$

Show that

$$\beta = \frac{2cq_0}{H_0(2q_0 - 1)^{\frac{3}{2}}}, \tag{F.7}$$

where q_0 is the present deceleration parameter.
Solution: The energy conservation equation in this case gives

$$\rho = \rho_0(a_0/a)^3. \tag{F.8}$$

Substituting this result to the Friedman equation we have

$$\dot{a}^2 = c^2\left(\gamma/a + 1\right), \ \text{ where } \ \gamma = 8\pi\rho_0 a_0^3/3c^2. \tag{F.9}$$

Then we calculate $\dot{a}^2$, using the parametric solution:

$$\dot{a}^2 = \left[\frac{d\left[\frac{\beta}{2}(\cosh\eta - 1)\right]}{d\left[\frac{\beta}{2c}(\sinh\eta - \eta)\right]}\right]^2 = c^2\frac{\sinh^2\eta}{(\cosh\eta - 1)^2} = c^2\frac{1 + \cosh\eta}{\cosh\eta - 1}. \tag{F.10}$$

Putting this into FE we have

$$c^2\frac{1 + \cosh\eta}{\cosh\eta - 1} = c^2\left(\frac{2\gamma}{\beta(\cosh\eta - 1)} + 1\right), \quad 1 + \cosh\eta = \frac{2\gamma}{\beta} + 1 + \cosh\eta. \tag{F.11}$$

We see that this parametric solution does satisfies the Friedman equation, if

$$\beta = \gamma = 8\pi\rho_0 a_0^3/3c^2. \tag{F.12}$$

From the Friedman equation, taken at the moment t_0 we have

$$H_0^2 a_0^2 = H_0^2\Omega_0 + c^2, \tag{F.13}$$

hence we can express a_0 in terms of H_0 and Ω_0 as

$$a_0 = \frac{c}{H_0\sqrt{1 - \Omega_0}}. \tag{F.14}$$

Then substituting this to the formula for β we have

$$\beta = \frac{c\Omega_0}{H_0(1 - \Omega_0)^{3/2}}. \tag{F.15}$$

5. Gravitational paradox

Problem: To illustrate that Newtonian theory is not applicable to the infinite Universe, show that gravitational forces from two halves of the homogeneous Universe are infinite and should destroy any body (a contradiction with an every day experience).
Solution: Dividing the Universe into infinite planes

$$|| \qquad\qquad \longleftarrow \bigcirc \longrightarrow \quad z \qquad\qquad ||$$

in cylindrical coordinates we have

$$dF_z(z) = Gm\rho dz\int_0^{2\pi} d\phi\int_0^{\infty}\frac{rdr\cos\theta}{a^2}. \tag{F.16}$$

Taking into account that

$$a = \sqrt{r^2 + z^2}, \quad \cos\theta = \frac{z}{a}, \tag{F.17}$$

we obtain

$$dF_z(z) = 2\pi G m \rho z dz \int_0^\infty \frac{r dr}{(r^2 + z^2)^{3/2}} = 2\pi G m \rho dz. \tag{F.18}$$

Hence,

$$F_z = \int_0^\infty dF(z) = 2\pi G m \rho \infty_0 dz = \infty. \tag{F.19}$$

6. Summary

(i) We obtained reasonable cosmological models predicting the future and explaining some global properties of the Universe in the past.
(ii) But we need GR to decide what is right and what is wrong in predictions made by Newtonian cosmological models.

III. SUPPLEMENTARY TO THE LECTURE 3

This is a brief introduction to General Relativity (GR)

A. The Principle of Equivalence and Geometrical Principle

1. *The Principle of equivalence*

The basic postulate of the General Relativity states that a uniform gravitational field is equivalent to (which means is not distinguishable from) a uniform acceleration. One cannot see locally the difference between standing on the surface of some gravitating body (for example the Earth) and moving in a rocket with corresponding acceleration. All bodies in given gravitational field move in the same manner, if initial conditions are the same. In other words, in given gravitational field all bodies move with the same acceleration. In absence of gravitational field, all bodies move also with the same acceleration relative to the non-inertial frame. Thus we can formulate **the Principle of Equivalence** which says: locally, any non-inertial frame of reference is equivalent to a certain gravitational field.
The important consequence of the Principle of Equivalence is that locally gravitational field can be eliminated by proper choice of the frame of reference. Such frames of reference are called **locally inertial or galilean frames of reference.** There is no experiment to distinguish between being weightless far out from gravitating bodies in space and being in free-fall in a gravitational field. Globally (not locally), "actual" Gravitational Fields can be distinguished from corresponding non-inertial frame of reference by its behavior at infinity: Gravitational Fields generated by gravitating bodies fall with distance.

2. *The Principle of equivalence in Newtonian theory*

In Newton's theory the motion of a test particle is determined by the following equation

$$m_{in}\vec{a} = -m_{gr}\nabla\phi, \tag{A.1}$$

where $\vec{a}$ is the acceleration of the test particle, ϕ is newtonian potential of gravitational field, m_{in} is the inertial mass of the test particle and m_{gr} is its gravitational mass (gravitational analog of the electric charge). The fundamental property of gravitational fields that all test particles move with the same acceleration for given ϕ is explained within frame of newtonian theory just by the following "coincidence":

$$\frac{m_{in}}{m_g} = 1, \tag{A.2}$$

i.e. inertial mass m_{in} is equal to gravitational mass m_{gr}.

3. *Summation convention*

The Einstein notation (introduced in 1916) or Einstein summation convention is a notational convention useful when dealing with long formulae of General relativity. According to this convention, when an index variable appears twice in a single term, once in an upper (superscript) and once in a lower (subscript) position, it implies that we are summing over all of its possible values. In this supplementary matirial , which is about the four dimensional space-time, the indices are 0,1,2,3 (0 represents the time coordinate). We will use the Roman alphabet for such indices. For spatial indices, 1,2,3, we will use the Greek alphabet. (In other textbooks Roman and Greek may be reversed.)
Example 1.

$$A^iB_i \equiv \sum_{i=0}^{3} A^iB_i = A^0B_0 + A^1B_1 + A^2B_2 + A^3B_3, \quad A^\alpha B_\alpha \equiv \sum_{i=1}^{3} A^\alpha B_\alpha = A^1B_1 + A^2B_2 + A^3B_3. \tag{A.3}$$

Example 2. The following expressions don't imply summation:

$$A_i B_i, \;\; A^i B^i, \;\; A_\alpha B_\alpha \;\; A^\alpha B^\alpha. \tag{A.4}$$

Free indices are indices which don't participate in summation, i.e. free indices can not appear twice in the same term. Notation and position of free indices in left and right side of equations should be the same.
Example 3. The following expressions are written properly:

$$A_i = B_i, \;\; A^i = B^i, \;\; A_\alpha = B_\alpha \;\; A^\alpha = B^\alpha. \tag{A.5}$$

The following expressions are wrong and have no sense:

$$A_i = B^i, \;\; A^i = B_i, \;\; A_n = B_i \;\; A^\alpha = B^\beta. \tag{A.6}$$

Repeating indices should appear twice as explained before and can not appear in the same term more than twice.
Example 4. The following expressions are written properly:

$$A^i B_i C_m P^m, \;\; A^i B^m C_m. \tag{A.7}$$

The following expressions are wrong and have no sense:

$$A^i B_i C_i P^m, \;\; A^i B_i C_i P^i, \;\; A_i = B^i, \;\; A^i = B_i, \;\; A_n = B_i \;\; A^\alpha = B^\beta. \tag{A.8}$$

Summation convention saves time and helps very much if you manipulate properly.
Example 5.: The following equation

$$A^i B_k C_m = P^v L_{vnj} G^{nj} Q^i_{km}, \tag{A.9}$$

corresponds to 64 equations with 64 terms on right hand side in each equation!!!

4. Gravity as a space-time geometry

Special Relativity is the theory which is valid only if we work within very special frames of reference (the global inertial frames). For such frames of reference the following combination of time and space coordinates remains invariant whatever global inertial frame of references is chosen

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2. \tag{A.10}$$

This combination is called the interval. All space-time coordinates in different global inertial frames of reference are related with each other by the Lorentz transformations which leave the shape of the interval unchanged. But this is not the case if one considers transformation of coordinates in more general case, when at least one of the two frames of reference is non-inertial. This interval is not reduced anymore to the simple sum of squares of the coordinate differentials and can be written in the following more general quadratic form:

$$ds^2 = g_{ik} dx^i dx^k \equiv \sum_{i=0}^{3} \sum_{k=0}^{3} g_{ik} dx^i dx^k, \tag{A.11}$$

where repeating indices mean summation. In inertial frames of reference

$$g_{00} = 1, \quad g_{11} = g_{22} = g_{33} = -1, \;\; \text{and} \;\; g_{ik} = 0, \quad \text{if} \quad i \neq k. \tag{A.12}$$

Example 6: Transformation to uniformly rotating frame.

$$x = x' \cos \Omega t - y' \sin \Omega t, \;\; y = x' \sin \Omega t + y' \cos \Omega t, \;\; z = z', \tag{A.13}$$

where Ω is the angular velocity of rotation around z-axis. In this non-inertial frame of reference

$$ds^2 = [c^2 - \Omega^2 (x'^2 + y'^2)] dt^2 - dx'^2 - dy'^2 - dz'^2 + 2\Omega y' dx' dt - 2\Omega x' dy' dt. \tag{A.14}$$

The fundamental physical concept of General Relativity is that gravitational field is identical to geometry of curved space-time. This idea, called **the Geometrical principle**, entirely determines the mathematical structure of General Relativity.
According to GR gravity is nothing but manifestation of space-time 4-geometry.

The geometry is determined by the interval

$$ds^2 = g_{ik}(x^m)dx^i dx^k, \tag{A.15}$$

where $g_{ik}(x^m)$ is called the metric tensor. What is exactly meant by the term "tensor" we will discuss later. At the present moment we can consider $g_{ik}(x^m)$ as a 4×4 -matrix and all its components in general case can depend on all 4 coordinates x^m , where $m = 0,\ 1,\ 2,\ 3$. All information about the geometry of space-time is contained in $g_{ik}(x^m)$. The dependence of $g_{ik}(x^m)$ on x^m means that this geometry is different in different events, which implies that the space-time is curved and its geometry is not Euclidian. Such sort of geometry is the the subject of mathematical discipline called Differential Geometry developed in XIX Century.
Thus, according to General Relativity gravitational field is identical to geometry of curved space-time (see **Fig.13.3.** and **Fig.13.4.** for examples of highly curved space-time. These pictures are also taken from the very interesting astronomical web-site of Nick Strobel.
The fundamental physical concept of General Relativity is to identity of the gravitational field and the geometry of curved space-time is called the **Geometrical principle.** This principle entirely predetermines the mathematical structure of General Relativity.

5. *Explanation of the principle of equivalence in GR*

The GR gives very simple and natural explanation of the Principle of Equivalence: **In curved space-time all bodies move along geodesics, that is why their world lines are the same in given gravitational field.** The situation is the same as in flat space-time when free particles move along straight lines which are geodesics in flat space-time.
The one of the main statements of General Theory of Relativity is the following: If we know g_{ik}, we can determine completely the motion of test particles and performance of all test fields. [Test particle or test field means that gravitational field generated by these test objects is negligible.] The metric tensor g_{ik} itself is determined by physical content of the space-time.
In any curved space-time (i.e in the actual gravitational field) there is no global galilean frames of reference. In flat space-time, if me work in non-inertial frames of reference metrics looks like the metric in gravitational field (because according to the Equivalence Principle **locally** actual gravitational field is not distinguishable from corresponding non-inertial frame of reference), nevertheless local galilean frames of reference do exist. The local galilean frame of reference is equivalent to the freely falling frame of reference in which locally gravitational field is eliminated. From geometrical point of view to eliminate gravitational field locally means to find such frame of reference in which

$$g_{ik} \to \eta_{ik} \equiv \mathrm{diag}(1, -1, -1, -1). \tag{A.16}$$

If space-time is flat and one works with inertial frames of reference the world lines of free particles are straight lines (another formulation of the first law of Newton). For particles moving with acceleration the world lines are curved (see
The fact that all bodies move with the same acceleration in given gravitational field seems to be now absolutely clear, because all bodies in given gravitational field moves along the same geodesics (the world lines which are the most close to the straight world lines). The shape of geodesics is determined only by geometry of space-time itself.

B. Physical Geometry of Space - Time

III B 1 Proper time,
III B 2 Spatial distance.

1. *Proper time*

One of the most central problems in the geometry of 4-spacetime can be formulated as follows. If the metric tensor is given, how actual (measurable) time and distances are related with coordinates x^0, x^1, x^2, x^3 chosen in arbitrary way.

Let us consider the world line of an observer who uses some clock to measure the actual or **proper time** $d\tau$ between two infinitesimally close events in the same place in space. Obviously we should put $dx^1 = dx^2 = dx^3 = 0$. Let us define proper time exactly as in Special Relativity:

$$d\tau = \frac{ds}{c}, \text{ then } ds^2 \equiv c^2 d\tau^2 = g_{ik} dx^i dx^k = g_{00}(dx^0)^2, \text{ thus } d\tau = \frac{1}{c}\sqrt{g_{00}}dx^0. \tag{B.1}$$

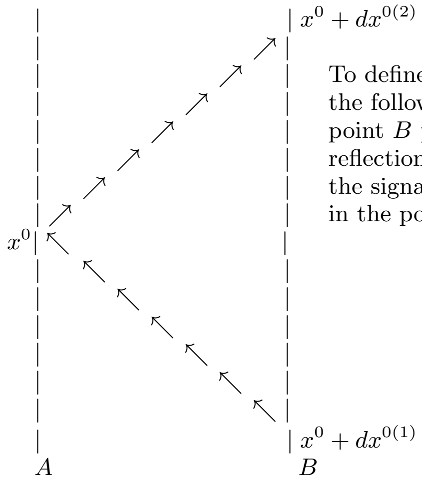
For the proper time between any two events (not necessary that these events are infinitesimally close) occurring at the same point in space we have

$$\tau = \frac{1}{c}\int \sqrt{g_{00}}dx^0. \tag{B.2}$$

2. *Spatial distance*

Separating the space and time coordinates in ds and considering $y = dx^0$ as a variable to find, we have

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta + 2g_{0\alpha}dx^0 dx^\alpha + g_{00}(dx^0)^2 = C + 2By + Ay^2, \text{ where } C = g_{\alpha\beta}dx^\alpha dx^\beta,\ B = g_{0\alpha}dx^\alpha,\ A = g_{00}. \tag{B.3}$$



To define dl we will use a light signal according to the following procedure: A light signal is emitted from some point B propagates to a point A and then after reflection at the moment corresponding to time coordinate x^0 the signal propagates back over the same path and is detected in the point B

The interval between the events which belong to the same world line of light in Special and General Relativity is always equal to zero: $ds = 0$**.** Hence

$$C + 2By + Ay^2 = 0. \tag{B.4}$$

Solving this quadratic equation with respect to $y = dx^0$ we find two roots:

$$dx^{0(1)} = \frac{-B - \sqrt{B^2 - AC}}{A} = \frac{1}{g_{00}}\left(-g_{0\alpha}dx^\alpha - \sqrt{(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00})dx^\alpha dx^\beta}\right) \tag{B.5}$$

$$dx^{0(2)} = \frac{-B + \sqrt{B^2 - AC}}{A} = \frac{1}{g_{00}}\left(-g_{0\alpha}dx^\alpha + \sqrt{(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00})dx^\alpha dx^\beta}\right) \tag{B.6}$$

$$dx^{0(2)} - dx^{0(1)} = \frac{2}{g_{00}}\sqrt{(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00})dx^\alpha dx^\beta}, \text{ then } dl = \frac{c}{2}d\tau = \frac{c}{2}\frac{\sqrt{g_{00}}}{c}(dx^{0(2)} - dx^{0(1)}), \tag{B.7}$$

$$\text{finally, } dl^2 = \gamma_{\alpha\beta}dx^\alpha dx^\beta, \text{ where } \gamma_{\alpha\beta} = -g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}}. \tag{B.8}$$

Thus, if we know g_{ik}, which is the pure geometrical object, we can determine proper time and physical spatial distance, using physical procedure of sending light signal. This is a really good example of relationship between Geometry and Physics.

C. The Principle of Covariance

III C 1 The Principle of covariance in Newtonian theory and in GR ,
III C 2 The Principle of equivalence in Newtonian theory.

1. *The Principle of covariance in Newtonian theory and in GR*

This Principle of Covariance says:
The shape of all physical equations should be the same in an arbitrary frame of reference. Otherwise the physical equations being different in gravitational field and in inertial frames of reference would have different solutions. In other words, these equations would predict the difference between gravitational field and non-inertial frame of reference and hence, would contradict to experimental data. This principle refers to the most general case of non-inertial frames (in contrast to the Special Theory of Relativity which works only in inertial frames of reference). Hence this principle is nothing but more mathematical formulation of the Principle of Equivalence that there is no way experimentally to discriminate between gravitational field and non-inertial frame of reference. This principle set very severe requirements to all physical equations and predetermines the mathematical structure of General Relativity: all equations should contain only tensors. By definition, tensors are objects which are transformed properly in the course of coordinate transformations from one frame of reference to another. Taking into account that non-inertial frames of reference in 4-dimensional space-time correspond to curvilinear coordinates, it is necessary to develop four-dimensional differential geometry in arbitrary curvilinear coordinates.

2. *Transformation of coordinates*

Let us consider the transformation of coordinates from one frame of reference (x^0, x^1, x^2, x^3) to another, ($x^{'0}$, $x^{'1}$, $x^{'2}$, $x^{'3}$):

$$x^0 = f^0(x'^0, x'^1, x'^2, x'^3), \;\; x^1 = f^1(x'^0, x'^1, x'^2, x'^3), \;\; x^2 = f^2(x'^0, x'^1, x'^2, x'^3), \;\; x^3 = f^1(x'^0, x'^1, x'^2, x'^3). \quad \text{(C.1)}$$

Then according to the summation convention

$$dx^i = \frac{\partial x^i}{\partial x'^k} dx'^k = S^i_k dx'^k, \;\; i, k = 0,\ 1,\ 2,\ 3, \quad \text{(C.2)}$$

where

$$S^i_k = \frac{\partial x^i}{\partial x'^k} \quad \text{(C.3)}$$

is a transformation matrix. Remember that all repeating indices mean summation, otherwise even such basic transformation would be written ugly. To demonstrate that summation convention is really very useful, I will write, the first and the last time, the same transformation without using the summation convention

$$dx^0 = \frac{\partial x^0}{\partial x'^0} dx'^0 + \frac{\partial x^0}{\partial x'^1} dx'^1 + \frac{\partial x^0}{\partial x'^2} dx'^2 + \frac{\partial x^0}{\partial x'^3} dx'^3 = S^0_0 dx'^0 + S^0_1 dx'^1 + S^0_2 dx'^2 + S^0_3 dx'^3,$$
$$dx^1 = \frac{\partial x^1}{\partial x'^0} dx'^0 + \frac{\partial x^1}{\partial x'^1} dx'^1 + \frac{\partial x^1}{\partial x'^2} dx'^2 + \frac{\partial x^1}{\partial x'^3} dx'^3 = S^1_0 dx'^0 + S^1_1 dx'^1 + S^1_2 dx'^2 + S^1_3 dx'^3,$$
$$dx^2 = \frac{\partial x^2}{\partial x'^0} dx'^0 + \frac{\partial x^2}{\partial x'^1} dx'^1 + \frac{\partial x^2}{\partial x'^2} dx'^2 + \frac{\partial x^2}{\partial x'^3} dx'^3 = S^2_0 dx'^0 + S^2_1 dx'^1 + S^2_2 dx'^2 + S^2_3 dx'^3,$$

$$dx^3 = \frac{\partial x^3}{\partial x'^0} dx'^0 + \frac{\partial x^3}{\partial x'^1} dx'^1 + \frac{\partial x^3}{\partial x'^2} dx'^2 + \frac{\partial x^3}{\partial x'^3} dx'^3 = S^3_0 dx'^0 + S^3_1 dx'^1 + S^3_2 dx'^2 + S^3_3 dx'^3. \quad \text{(C.4)}$$

D. Tensors

III D 1 Scalars and vectors,
III D 2 Tensors of arbitrary ranks,
III D 3 Reciprocal tensors.

1. Scalars and vectors

Now we can give the definition of the Contravariant four-vector: **The Contravariant four-vector is the combination of four quantities (components) A^i, which are transformed like differentials of coordinates:**

$$A^i = S^i_k A'^k. \tag{D.1}$$

Let φ is scalar field, then

$$\frac{\partial\varphi}{\partial x^i} = \frac{\partial\varphi}{\partial x'^k}\frac{\partial x'^k}{\partial x^i} = \tilde{S}^k_i \frac{\partial\varphi}{\partial x'^k}, \tag{D.2}$$

where $\tilde{S}^k_i$ is another transformation matric. What is relation of this matrix with the previous transformation matrix S^i_k? If we take product of these matrices, we obtain

$$S^i_n \tilde{S}^n_k = \frac{\partial x^i}{\partial x'^n}\frac{\partial x'^n}{\partial x^k} = \frac{\partial x^i}{\partial x^k} = \delta^i_k, \tag{D.3}$$

where δ^i_k is so called **Kronneker symbol**, which actually is nothing but the unit matrix:

$$\delta^i_k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{D.4}$$

In other words $\tilde{S}^i_k$ is inverse or reciprocal with respect to S^i_k.
Now we can give the definition of the covariant four-vector: **The Covariant four-vector** is the combination of four quantities (components) A_i, which are transformed like like components of the gradient of a scalar field:

$$A_i = \frac{\partial x'^k}{\partial x^i} A'_k. \tag{D.5}$$

Note, that for contravariant vectors we always use upper indices, which are called **contravariant indices**, while for covariant vectors we use low indices, which are called **covariant indices.** In GR summation convention always means that on of the two repeating indices should be contravariant and another should be covariant.
Example 7:

$$A^i B_i = A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3 \tag{D.6}$$

is the scalar product, while there is no summation if both indices are, say, covariant, for example:

$$A_i B_i = \begin{matrix} A_0 B_0, \text{ if } i = 0 \\ A_1 B_1, \text{ if } i = 1 \\ A_2 B_2, \text{ if } i = 2 \\ A_3 B_3, \text{ if } i = 3 \end{matrix} \tag{D.7}$$

2. Tensors of arbitrary ranks

Now we can generalize the definitions of vectors and introduce **tensors** entirely in terms of transformation laws. **Scalar** is the tensor of the 0 rank. It has only $4^{\mathbf{0}} = 1$ component and 0 number of indices. Transformation law is

$$A = A', \tag{D.8}$$

we see that transformation matrices appear in transformation law 0 times.
Contravariant and covariant vectors are tensors of the 1 rank. They have $4^{\mathbf{1}} = 4$ components and 1 index. Corresponding transformation laws are

$$A^i = S^i_n A'^n, \;\; A_i = \tilde{S}^n_i A'_n, \tag{D.9}$$

we see **1** transformation matrix in each transformation law.
Contravariant tensor of the 2 **rank** has $4^{\mathbf{2}} = 16$ components and 2 contravariant indices. Corresponding transformation law is

$$A^{ik} = S^i_n S^k_m A'^{nm}, \tag{D.10}$$

we see **2** transformation matrices in the transformation law.
Covariant tensor of the 2 **rank** has also $4^{\mathbf{2}} = 16$ components and 2 covariant indices. Corresponding transformation law is

$$A_{ik} = \tilde{S}^n_i \tilde{S}^m_k A'_{nm}, \tag{D.11}$$

we see **2** transformation matrices in the transformation law.
Mixed tensor of the 2 **rank** has $4^{\mathbf{2}} = 16$ components and 2 indices, 1 contravariant and 1 covariant. Corresponding transformation law is

$$A^i_k = S^i_n \tilde{S}^m_k A'^n_m, \tag{D.12}$$

we see **2** transformation matrices in the transformation law.
Covariant tensor of the 3 **rank** has $4^{\mathbf{3}} = 64$ components and 3 covariant indices. Corresponding transformation law is...
Mixed tensor of the $N + M$ **rank with** N **contravariant and** M **covariant indices** has $4^{\mathbf{N+M}} = 2^{2(N+M)}$ components and $N + M$ indices. Corresponding transformation law is

$$A^{i_1\; i_2\; ...\; i_N}_{k_1\; k_2\; ...\; k_M} = S^{i_1}_{n_1} S^{i_2}_{n_2} ... S^{i_N}_{n_N} \tilde{S}^{m_1}_{k_1} \tilde{S}^{m_2}_{k_2} ... \tilde{S}^{m_M}_{k_M} A'^{n_1\; n_2\; ...\; n_N}_{m_1 m_2 ... m_M}, \tag{D.13}$$

we see **N+M** transformation matrices in the transformation law.

3. Reciprocal tensors

Two tensors A_{ik} and B^{ik} are called reciprocal to each other if

$$A_{ik} B^{kl} = \delta^l_i. \tag{D.14}$$

We can introduce now a contravariant metric tensor g^{ik} which is reciprocal to the covariant metric tensor g_{ik}:

$$g_{ik} g^{kl} = \delta^l_i. \tag{D.15}$$

With the help of the metric tensor and its reciprocal we can form contravariant tensor from covariant tensors and vice versa, for example:

$$A^i = g^{ik} A_k, \quad A_i = g_{ik} A^k, \tag{D.16}$$

in other words we can rise and descend indices as we like, some sort of juggling with indices. We can say that contravariant, covariant and mixed tensors can be considered as different representations of the same geometrical object.

For the contravariant metric tensor itself we have very important representation in terms of the transformation matrix from locally inertial frame of reference (galilean frame) to an arbitrary non-inertial frame, let us denote it as $S^i_{(0)k}$. We know that in the galilean frame of reference

$$g^{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv \eta^{ik} \equiv \mathrm{diag}(1,-1,-1,-1), \tag{D.17}$$

hence

$$g^{ik} = S^i_{(0)n} S^k_{(0)m} \eta^{nm} = S^i_{(0)0} S^k_{(0)0} - S^i_{(0)1} S^k_{(0)1} - S^i_{(0)2} S^k_{(0)2} - S^i_{(0)3} S^k_{(0)3}. \tag{D.18}$$

This means that if we know the transformation law from the local galilean frame of reference to an arbitrary frame of reference, we know the metric at this arbitrary frame of reference and, hence, gravitational field, i.e. the geometry of space-time!!!

E. Covariant differentiation

1. *Parallel transport*

In Special Relativity if A_i is a vector dA^i is also a vector (the same is valid for any tensor). But in curvilinear coordinates this is not the case:

$$A_i = \frac{\partial x'^k}{\partial x^i} A'_k \;\; dA_i = \frac{\partial x'^k}{\partial x^i} dA'_k + A'_k \frac{\partial^2 x'^k}{\partial x^i \partial x^l} dx^l, \tag{E.1}$$

thus dA_i is not a vector unless x'^k are linear functions of x^k (like in the case of Lorentz transformations). Let us introduce another very useful notation:

$$, i = \frac{\partial}{\partial x^i} \tag{E.2}$$

According to the principle of covariance we can not afford to have not tensors in any physical equations, thus we should replace all differentials like

$$dA_i \;\text{ and }\; \frac{\partial A_i}{\partial x^k} \equiv A_{i,k} \tag{E.3}$$

by some corrected values which we will denote as

$$DA_i \;\text{ and }\; A_{i;k} \tag{E.4}$$

correspondingly. In arbitrary coordinates to obtain a differential of a vector which forms a vector we should subtract vectors in the same point, not in different as we have done before. Hence we need produce **a parallel transport or a parallel translation.** Under a parallel translation of a vector in galilean frame of reference its component don't change, but in curvilinear coordinates they do and we should introduce some corrections:

$$DA^i = dA^i - \delta A^i. \tag{E.5}$$

These corrections obviously should be linear with respect to all components of A_i and independently they should be linear with respect of dx^k, hence we can write these corrections as

$$\delta A^i = -\Gamma^i_{kl} A^k dx^l, \tag{E.6}$$

where Γ^i_{kl} **are called Christoffel Symbols** which obviously don't form any tensor, because DA_i is the tensor while as we know dA_i is not a tensor.

2. *Covariant derivatives and Christoffel symbols*

In terms of the Christoffel symbols

$$DA^i = (\frac{\partial A^i}{\partial x^l} + \Gamma^i_{kl} A^k) dx^l = (A^i_{,l} + \Gamma^i_{kl} A^k) dx^l, \;\; DA_i = (\frac{\partial A_i}{\partial x^l} - \Gamma^k_{il} A_k) dx^l = (A_{i,l} - \Gamma^k_{il} A_k) dx^l, \tag{E.7}$$

$$A^i_{;l} = \frac{\partial A^i}{\partial x^l} + \Gamma^i_{kl} A^k = A^i_{,l} + \Gamma^i_{kl} A^k, \;\; A_{i;l} = \frac{\partial A_i}{\partial x^l} - \Gamma^k_{il} A^k = A_{i,l} - \Gamma^k_{il} A^k. \tag{E.8}$$

To calculate the covariant derivative of tensor start with contravariant tensor which can be presented as a product of two contravariant vectors $A^i B^k$. In this case the corrections under parallel transport are

$$\delta(A^i B^k) = A^i \delta B^k + B^k \delta A^i = -A^i \Gamma^k_{lm} B^l dx^m - B^k \Gamma^i_{lm} A^l dx^n, \tag{E.9}$$

since these corrections are linear we have the same for arbitrary tensor A^{ik}:

$$\delta A^{ik} = -(A^{im} \Gamma^k_{ml} + A^{mk} \Gamma^i_{ml}) dx^l \;\; DA^{ik} = dA^{ik} - \delta A^{ik} \equiv A^{ik}_{;\,l} dx^l, \tag{E.10}$$

hence

$$A^{ik}_{;\,l} = A^{ik}_{,l} + \Gamma^i_{ml} A^{mk} + \Gamma^k_{ml} A^{im}. \tag{E.11}$$

In similar way we can obtain that

$$A^i_{k;\,l} = A^i_{k,l} - \Gamma^m_{kl} A^i_m + \Gamma^i_{ml} A^m_k, \;\; \text{and} \;\; A_{ik;\,l} = A_{ik,l} - \Gamma^m_{il} A_{mk} - \Gamma^m_{kl} A_{m,\,i}. \tag{E.12}$$

In the most general case when we have tensor of $m+n$ rank with m contravariant and n covariant indices the rule for calculation of the covariant derivative with respect to index p is the following

$$A^{i_1\, i_2\, \ldots\, i_m}_{j_1\, j_2\, \ldots\, j_n\, ;\, \mathbf{p}} = A^{i_1\, i_2\, \ldots\, i_m}_{j_1\, j_2\, \ldots\, j_n\, ,\, \mathbf{p}} + \Gamma^{\mathbf{i_1}}_{\mathbf{kp}} A^{\mathbf{k}\, i_2\, \ldots\, i_m}_{j_1\, j_2\, \ldots\, j_n} + \Gamma^{\mathbf{i_2}}_{\mathbf{kp}} A^{i_1\, \mathbf{k}\, \ldots\, i_m}_{j_1\, j_2\, \ldots\, j_n} + \;\ldots\; + \Gamma^{\mathbf{i}_m}_{\mathbf{kp}} A^{i_1\, i_2\, \ldots\, \mathbf{k}}_{j_1\, j_2\, \ldots\, j_n} -$$

$$- \Gamma^{\mathbf{k}}_{\mathbf{j_1}\, \mathbf{p}} A^{i_1\, i_2\, \ldots\, i_m}_{\mathbf{k}\, j_2\, \ldots\, j_n} - \Gamma^{\mathbf{k}}_{\mathbf{j_2}\, \mathbf{p}} A^{i_1\, i_2\, \ldots\, i_m}_{j_1\, \mathbf{k}\, \ldots\, j_n} - \;\ldots\; - \Gamma^{\mathbf{k}}_{\mathbf{j}_n\, \mathbf{p}} A^{i_1\, i_2\, \ldots\, i_m}_{j_1\, j_2\, \ldots\, \mathbf{k}}. \tag{E.13}$$

3. *The Relation of the Christoffel symbols to the metric tensor*

So far we don't know how the Christoffel symbols depend on coordinates, however we can prove that they are symmetric in the subscripts. Let some covariant vector A_i is the gradient of a scalar ϕ, i.e. $A_i = \phi_{,i}$. Then

$$A_{k;\,i} - A_{i;\,k} = \phi_{,k,i} - \Gamma^l_{ki} \phi_{,l} - \phi_{,i,k} + \Gamma^l_{ik} \phi_{,l} = \left(\Gamma^l_{ik} - \Gamma^l_{ki}\right) \phi_{,l}. \tag{E.14}$$

In Galilean coordinates

$$\Gamma^l_{ik} = \Gamma^l_{ki} = 0, \;\; \text{hence} \;\; A_{k;\,i} - A_{i;\,k} = 0. \tag{E.15}$$

Taking into account that $A_{k;\,i} - A_{i;\,k}$ is a tensor we conclude that if it equals to zero in one system of coordinates it should be equal to zero in any other coordinate system, hence

$$\Gamma^l_{ik} = \Gamma^l_{ki} \;\; \text{in any coordinate system.} \tag{E.16}$$

This is a typical example of the proof widely used in General Relativity: **If some equality between tensors is valid in one coordinate system then this equality is valid in arbitrary coordinate system. This is obvious advantage to deal with tensors.** Then we can show that covariant derivatives of g_{ik} are equal to zero. Indeed:

$$DA_i = g_{ik}DA^k \;\; DA_i = D(g_{ik}A^k) = g_{ik}DA^k + A^k Dg_{ik}, \;\; \text{hence} \;\; g_{ik}DA^k = g_{ik}DA^k + A^k Dg_{ik}, \tag{E.17}$$

which obviously means that

$$A^k Dg_{ik} = 0. \tag{E.18}$$

Taking into account that A^k is arbitrary vector, we conclude that

$$Dg_{ik} = 0. \tag{E.19}$$

This is another example of proof in General Relativity: **If the the sum $B_{ik}A^i = 0$ for arbitrary vector A^i then the tensor $B_{ik} = 0$.** Then taking into account that

$$Dg_{ik} = g_{ik;m}dx^m = 0 \tag{E.20}$$

for arbitrary infinitesimally small vector dx^m we have

$$g_{ik;m} = 0. \tag{E.21}$$

Now we are ready to relate the Christoffel symbols to the metric tensor. Introducing useful notation

$$\Gamma_{k,\, il} = g_{km}\Gamma^m_{il}, \tag{E.22}$$

we have

$$g_{ik;\, l} = \frac{\partial g_{ik}}{\partial x^l} - g_{mk}\Gamma^m_{il} - g_{im}\Gamma^m_{kl} = \frac{\partial g_{ik}}{\partial x^l} - \Gamma_{k,\, il} - \Gamma_{i,\, kl} = 0. \tag{E.23}$$

Permuting the indices i, k and l twice as

$$i \to k, \;\; k \to l, \;\; l \to i, \tag{E.24}$$

we have

$$\frac{\partial g_{ik}}{\partial x^l} = \Gamma_{k,\, il} + \Gamma_{i,\, kl}, \quad \frac{\partial g_{li}}{\partial x^k} = \Gamma_{i,\, kl} + \Gamma_{l,\, ik} \;\; \text{and} \;\; -\frac{\partial g_{kl}}{\partial x^i} = -\Gamma_{l,\, ki} - \Gamma_{k,\, li}. \tag{E.25}$$

Taking into account that

$$\Gamma_{k,\, il} = \Gamma_{k,\, li}, \tag{E.26}$$

after summation of these three equation we have

$$g_{ik,l} + g_{li,k} - g_{kl,i} = 2\Gamma_{i,\, kl} \tag{E.27}$$

Finally

$$\Gamma^i_{kl} = \frac{1}{2}g^{im}\left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m}\right). \tag{E.28}$$

Now we have expressions for the Christoffel symbols in terms of the metric tensor and hence we know their dependence on coordinates.

F. Physical Applications

The previous material can be summarized as follows:
Gravity is equivalent to curved space-time, hence in all differentials of tensors we should take into account the change in the components of a tensor under an infinitesimal parallel transport. Corresponding corrections are expressed in terms of the Cristoffel symbols and reduced to replacement of any partial derivative by corresponding covariant derivative. In other words we can say that if one wants to take into account all effects of Gravity on any local physical process, described by the corresponding equations written in framework of Special Relativity, one should just replace all partial derivatives by covariant derivatives in these equation according to the following very nice and simple but actually very strong and important formulae:

$$\mathbf{d} \rightarrow \mathbf{D} \ \ \textbf{and} \ \ , \rightarrow ; \tag{F.1}$$

Example 8: In special Relativity the following is obviously true

$$dg_{ik} = 0 \ \text{ and } \ g_{ik,l} = 0, \tag{F.2}$$

while in General Relativity

$$Dg_{ik} = 0 \ \text{ and } \ g_{ik;l} = 0. \tag{F.3}$$

Example 9: Let us apply above formulae to description of motion of a test particle in a given gravitational field. Let

$$u^i = \frac{dx^i}{ds} \tag{F.4}$$

is the four-velocity. Then equation for motion of a free particle in absence of gravitational field is

$$\frac{du^i}{ds} = 0. \tag{F.5}$$

In presence of a gravitational field this equation is generalized to the equation

$$\frac{Du^i}{ds} = 0, \tag{F.6}$$

which gives

$$\frac{Du^i}{ds} = \frac{du^i}{ds} + \Gamma^i_{kn} u^k \frac{dx^n}{ds} = \frac{d^2x^i}{ds^2} + \Gamma^i_{kn} u^k u^n = 0. \tag{F.7}$$

Thus from the physical point of view the equation (F.7)describes the motion of free particle in a given gravitational field and

$$\frac{d^2x^i}{ds^2} = -\Gamma^i_{kl} \frac{dx^k}{ds} \frac{dx^l}{ds} \tag{F.8}$$

is the four-acceleration, while from geometrical point of view this equation is the equation for geodesics in a curved space-time. That is why all particles move along geodesics, hence with the same acceleration. Now this experimental fact is not coincidence anymore but consequence of geometrical interpretation of gravity.

G. The Riemann curvature tensor

1. Differentiating twice

We know that $A_{i,k,l}-A_{i,l,k}=0$. What can we say about the commutator $A_{i;\,k;\,l}-A_{i;\,l;\,k}$? Straightforward calculations show that this is not equal to zero in the presence of gravitational field and can be presented as

$$A_{i;\,k;\,l}-A_{i;\,l;\,k}=A_mR^m_{ikl},\ \text{ where }\ R^i_{klm}=\Gamma^i_{km,l}-\Gamma^i_{kl,m}+\Gamma^i_{nl}\Gamma^n_{km}-\Gamma^i_{nm}\Gamma^n_{kl}. \tag{G.1}$$

The object R^i_{klm} obviously is a tensor and called **the curvature Riemann tensor.** We know that if at least one component of a tensor is not equal to zero at least in one frame of reference, the same is true for any other frame of reference, in other words, tensors can not be eliminated by transformations of coordinates. The Riemann tensor describes actual tidal gravitational field, which is not local and, hence, can not be eliminated even in the locally inertial frame of reference. Let us calculate the curvature Riemann tensor directly:

$$A_{i;k;l}-A_{i;l;k}=A_{i;k,l}-\Gamma^m_{li}A_{m;k}-\Gamma^m_{lk}A_{i;m}-A_{i;l,k}+\Gamma^m_{ki}A_{m;l}+\Gamma^m_{kl}A_{i;m}=$$

$$\left(A_{i,k}-\Gamma^m_{ik}A_m\right)_{,l}-\Gamma^m_{li}\left(A_{m,k}-\Gamma^n_{mk}A_n\right)-\left(A_{i,l}-\Gamma^m_{il}A_m\right)_{,k}+\Gamma^m_{ki}\left(A_{m,l}-\Gamma^n_{ml}A_n\right)=$$

$$A_{i,k,l}-A_{i,l,k}-\Gamma^m_{ik}A_{m,l}-\Gamma^m_{il}A_{m,k}-\Gamma^m_{kl}A_{i,m}+\Gamma^m_{il}A_{m,k}+\Gamma^m_{ik}A_{m,l}+\Gamma^m_{lk}A_{i,m}-$$

$$-\Gamma^m_{ik,l}A_m+\Gamma^m_{il}\Gamma^p_{mk}A_p+\Gamma^m_{kl}\Gamma^p_{im}A_p+\Gamma^m_{ik,l}A_m-\Gamma^m_{ik}\Gamma^p_{ml}A_p-\Gamma^m_{lk}\Gamma^p_{im}A_p=$$

$$=A_m\left(-\Gamma^m_{ik,l}+\Gamma^p_{il}\Gamma^m_{pk}+\Gamma^p_{kl}\Gamma^m_{ip}+\Gamma^m_{il,k}-\Gamma^p_{ik}\Gamma^m_{pl}-\Gamma^p_{lk}\Gamma^m_{ip}\right)=A_m\left(-\Gamma^m_{ik,l}+\Gamma^p_{il}\Gamma^m_{pk}+\Gamma^m_{il,k}-\Gamma^p_{ik}\Gamma^m_{pl}\right). \tag{G.2}$$

Thus

$$R^m_{ikl}=\Gamma^m_{il,k}-\Gamma^m_{ik,l}+\Gamma^p_{il}\Gamma^m_{pk}-\Gamma^p_{ik}\Gamma^m_{pl}. \tag{G.3}$$

With the help of the Riemann tensor we can differentiate twice tensors of arbitrary rank, for example,

$$A_{i;\,k;\,l}-A_{i;\,l;\,k}=A_mR^m_{ikl}\quad A_{ik;\,l;\,m}-A_{ik;\,m;\,l}=A_{in}R^n_{klm}+A_{nk}R^n_{ilm}. \tag{G.4}$$

The Riemann curvature tensor appears in so called the geodesic deviation equation. This equation measures the change in separation of neighboring geodesics. In the language of mechanics it measures the rate of relative acceleration of two particles moving forward on neighboring geodesics separated by a 4-vector η^i:

$$\frac{d^2\eta^i}{ds^2}=R^i_{klm}u^ku^l\eta^m,\ \text{ where }\ u^i=\frac{dx^i}{ds}. \tag{G.5}$$

If gravitational field is weak and all motions are slow, $u^i\approx\delta^i_0$, the above equation is reduced to the Newtonian equation for the tidal acceleration.

2. *Properties of the Riemann curvature tensor*

Let us introduce covariant version of the Riemann tensor

$$R_{iklm} = g_{in} R^n_{klm}. \tag{G.6}$$

One can easily show that

$$R_{iklm} = \frac{1}{2}\left(g_{im,k,l} + g_{kl,i,m} - g_{il,k,m} - g_{km,i,l}\right) + g_{np}\left(\Gamma^n_{kl}\Gamma^p_{im} - \Gamma^n_{km}\Gamma^p_{il}\right). \tag{G.7}$$

The following properties of the curvature tensor R_{iklm} are important for derivation of EFEs:

$$1)\;\; R_{iklm} = -R_{kilm} = -R_{ikml},\;\; 2)\;\; R_{iklm} = R_{lmik},\;\; 3)\;\; R_{iklm} + R_{imkl} + R_{ilmk} = 0 \tag{G.8}$$

$$\text{and }\; 4)\;\; R^n_{ikl;\,m} + R^n_{imk;\,l} + R^n_{ilm;\,k} = 0. \tag{G.9}$$

The property 4) is called Bianchy identity and can be proofed in locally-geodesic coordinate system,

$$\text{where }\; \Gamma^i_{kl} = 0 \;\text{ and }\; R^n_{ikl;m} = R^n_{ikl,m} = \Gamma^n_{il,m,k} - \Gamma^n_{ik,m,l}. \tag{G.10}$$

H. Einstein Field Equations(EFEs)

1. *The Stress-Energy Tensor (SET)*

In general Relativity, as well as in all other parts of Physics, **the stress-energy tensor** is symmetric and contains ten independent components (see Table 1 below).

	Table 1	
Component	What it represents	Number of components
T_{00}	the energy density	1
$T_{0\alpha}$ $T_{\alpha 0} \equiv T_{0\alpha}$	the flux of energy the density of the momentum	3
$T_{\alpha\alpha}$	the pressure (if it positive) or tension (if it negative)	3
$T_{\alpha\beta}, \alpha \neq \beta$	shear stress	3

All these ten components participate in generation of gravitational field, while in Newton gravity the only source of gravitational field is the mass density.
In cosmology we study the nearly isotropic Universe, in this case all out of diagonal components are equal to zero and we have

$$T_{ik} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{H.1}$$

where ϵ is the energy density and P is a pressure or tension. In cosmology these two quantities are related be the equation called **the equation of state:**

$$p = \alpha\epsilon, \tag{H.2}$$

where $\alpha = \gamma - 1$ is a constant called along with γ **the equation of state parameter.**
Different substances are described by the different values of parameter α and γ in (H.2) (see Table 2 below).

	Table 2	
α	γ	Substance
0	1	Dust
1/3	4/3	CMB
$-1/3$	2/3	Cosmic strings (not examinable)
$-1 < \alpha < -1/3$	$0 < \gamma < 2/3$	Dark energy
-1	0	Dark energy in form of the Λ-term
$\alpha < -1$	$\gamma < 0$	Fantom energy (not examinable any way)

2. *Heuristics derivation of EFEs*

It seems to be a good idea to relate the curvature Riemann tensor tensor with the stress-energy tensor? Unfortunately the rang of this tensor is 4, which is too big in comparison with rang 2 for the stress-energy tensor. To solve this problem we can construct the tensor of the second rank. Replacing k by l and l by k and then just putting $m = l$ we obtain a tensor of second rank, called the Ricci tensor:

$$R_{ik} = g^{lm} R_{limk} = R^l_{ilk} \tag{H.3}$$

$$R_{ik} = \Gamma^l_{ik,l} - \Gamma^l_{il,k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km}. \tag{H.4}$$

It is easy to see that

$$R_{ik} = R_{ki}. \tag{H.5}$$

We can produce further construction to obtain a zero rank tensor, i.e. a scalar

$$R = g^{ik} R_{ik} = g^{il} g^{km} R_{iklm}, \tag{H.6}$$

which is called **the scalar curvature.** Einstein (with the help of Gilbert) introduced the following tensor

$$G_{ik} = R_{ik} - \frac{1}{2} g_{ik} R, \tag{H.7}$$

which is called **the Einstein'tensor.** Now **the Einstein Field Equations (EFEs)** can be written as

$$G_{ik} = \kappa T_{ik}, \tag{H.8}$$

where T_{ik} is the stress-energy tensor (sometimes stress-energy-momentum tensor), which describes the density and flux of energy and momentum. The constant κ in EFE is called **the Einstein constant**. To determine this constant we can use so called **the correspondence principle**, which states that the EFEs in weak-field and slow-motion approximation should reduce to Newton's law of gravity, described by the Poisson's equation

$$\Delta\phi = 4\pi G\rho \tag{H.9}$$

According to this obvious principle

$$\kappa = \frac{8\pi G}{c^4}. \tag{H.10}$$

Thus the EFEs can be written as

$$\mathbf{R_{ik} - \frac{1}{2} g_{ik} R = \frac{8\pi G}{c^4} T_{ik}}. \tag{H.11}$$

Despite the simple appearance of the equations they are, in fact, quite complicated. The EFEs are 10 equations for 10 independent components of the metric tensor g_{ik}. Taking into account that the Ricci, R_{ik}, as well as the scalar curvature, R, contain linear combinations of second partial derivatives of the metric tensor and nonlinear combinations of its first derivatives, we can see thatthe EFEs is a system of 10 coupled, nonlinear, hyperbolic-elliptic partial differential equations.
As follows from the Bianchi identity

$$R^l_{m;\,l} = \frac{1}{2} R, m, \tag{H.12}$$

hence

$$T^i_{k;i} = 0, \tag{H.13}$$

which means that according to the EFEs the covariant divergence of the stress-energy tensor is equal to zero, i.e. the EFEs contain all conservation laws which from mathematical point of view are equivalent to all dynamical equations for matter and fields. In order words, the EFEs describe in self-consistent way the distribution and motion of matter and fields in curved space-time, while the geometry of the space-time itself is determined trough the EFEs by the distribution and the motion of matter and fields.

I. Examples, Problems and Summary

1. *Transformations of tensors*

Problem: a) Give the definition of the mixed tensor of the second rank, A^i_k, and b) the mixed tensor of the third rank, B^i_{km}. c) Given that in the local Galilean frame $x^i_{[G]}$ of reference a mixed tensor of the fourth rank, C^{ik}_{lm} has the only one non-vanishing component, $C^{00}_{00[G]} = 1$, and all other components are equal to zero. Write down all components of this mixed tensor in arbitrary frame of reference, x^i, in terms of the transformation matrices $S^l_{m[G]} = \frac{\partial x^l}{\partial x^m_{[G]}}$ and $\tilde{S}^l_{m[G]} = \frac{\partial x^l_{[G]}}{\partial x^m}$.
Solutions: a) The mixed tensor of the second rank is the object containing $4^2 = 16$ components A_{ik} which in the course of an arbitrary transformation from one frame of reference, x'^n, to another, x^m, are transformed according to the following transformation law:

$$A^i_k = S^i_v \tilde{S}^u_k A'_{vu}, \text{ where } \tilde{S}^l_m = \frac{\partial x'^l}{\partial x^m}. \tag{I.1}$$

b) The mixed tensor of the third rank with one contravariant and two covariant indices is the object containing $4^3 = 64$ components B^i_{km} which in the course of an arbitrary transformation from one frame of reference, x'^n, to another, x^m, are transformed according to the following transformation law:
c) The law of transformation for the tensor C^{ik}_{lm} from local Galilean to arbitrary frame of reference is

$$C^{ik}_{lm} = S^i_p S^k_v \tilde{S}^u_l \tilde{S}^w_m C^{pv}_{uw(G)}, \text{ where } S^l_m = S^l_{(G)m} = \frac{\partial x^l}{\partial x^m_G}. \tag{I.2}$$

It is given that

$$C^{pv}_{uw(G)} = \delta^p_0 \delta^v_0 \delta^1_u \delta^1_w, \text{ hence } C^{ik}_{lm} = S^i_0 S^k_0 \tilde{S}^1_l \tilde{S}^1_m. \tag{I.3}$$

2. *The Ricci tensor and SET*

Problem: Using the EFEs and Bianchi identity (see rubric) show that the stress-energy tensor satisfies conservation law $T^i_{k;i} = 0$.
Solution: Let us contract the Biancci identity over indices i and n (i.e. produce summation $i = n$).

$$R^i_{klm;n} + R^i_{knl;m} + R^i_{kmn;l} = 0 \;\Rightarrow\; R^i_{klm;i} + R^i_{kil;m} + R^i_{kmi;l} = 0. \tag{I.4}$$

According to the definition of the Ricci tensor

$$R^i_{kil} = R_{kl}, \text{ hence the second term in (I.4) can be rewritten as } R^i_{kil;m} = R_{kl;m}. \tag{I.5}$$

Taking into account that the Riemann tensor is antisymmetric with respect permutations of indices within the same pair

$$R^i_{kmi} = -R^i_{kim} = -R_{km}, \tag{I.6}$$

the third term can be rewritten as

$$R^i_{kmi;l} = -R_{km;l}. \tag{I.7}$$

The first term can be rewritten as

$$R^i_{klm;i} = g^{ip} R_{pklm;i}, \tag{I.8}$$

then taking mentioned above permutation twice we can rewrite the first term as

$$R^i_{klm;i} = g^{ip} R_{pklm;i} = -g^{ip} R_{kplm;i} = g^{ip} R_{kpml;i}. \tag{I.9}$$

After all these manipulations we have

$$g^{ip} R_{kpml;i} + R_{kl;m} - R_{km;l} = 0. \tag{I.10}$$

Then multiplying by g^{km} and taking into account that all covariant derivatives of the metric tensor are equal to zero, we have

$$g^{km} g^{ip} R_{kpml;i} + g^{km} R_{kl;m} - g^{km} R_{km;l} = \left(g^{km} g^{ip} R_{kpml}\right)_{;i} + \left(g^{km} R_{kl}\right)_{;m} - \left(g^{km} R_{km}\right)_{;l} = 0. \tag{I.11}$$

In the first term expression in brackets can be simplified as

$$g^{km} g^{ip} R_{kpml} = g^{ip} R_{pl} = R^i_l. \tag{I.12}$$

In the second term expression in brackets can be simplified as

$$g^{km} R_{kl} = R^m_l. \tag{I.13}$$

According to the definition of scalar curvature

$$R = g^{km} R_{km}, \tag{I.14}$$

the third term can be simplified as

$$\left(g^{km} R_{km}\right)_{;l} = R_{;l} = R_{,l}. \tag{I.15}$$

Thus

$$R^i_{l;i} + R^m_{l;m} - R_{,l} = 0, \tag{I.16}$$

replacing in the second term index of summation m by i we finally obtain

$$2R^i_{l;i} - R_{,l} = 0, \text{ or } R^i_{l;i} - \frac{1}{2} R_{,l} = 0. \tag{I.17}$$

Using the EFEs in the mixed form

$$R^i_k - \frac{1}{2}\delta^i_k R = \frac{8\pi G}{c^4} T^i_k, \tag{I.18}$$

we obtain

$$T^i_{k;i} = \frac{c^4}{8\pi G}\left(R^i_k - \frac{1}{2}\delta^i_k R\right)_{;i} = \frac{c^4}{8\pi G}\left(R^i_{k;i} - \frac{1}{2}\delta^i_k R_{,i}\right) = \frac{c^4}{8\pi G}\left(R^i_{k;i} - \frac{1}{2} R_{,k}\right) = 0. \tag{I.19}$$

3. *The scalar curvature and the trace of SET*

Problem: a) Show that the covariant divergence of the stress-energy tensor is equal to zero, $T^i_{k\,;\,k} = 0$. b) Th e stress-energy tensor has the following form

$$T_{ik} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}, \tag{I.20}$$

where ε is energy density and P is pressure (if $P > 0$) or tension (if $P < 0$). Using the Einstein equations express the scalar curvature in terms of ε and P.
Solution: a) Multiplying the EFEs

$$R_{ik} - \frac{1}{2} g_{ik} R = \frac{8\pi G}{c^4} T_{ik} \tag{I.21}$$

by g^{mk} we obtain

$$R^m_i - \frac{1}{2}\delta^m_i R = \frac{8\pi G}{c^4} T^m_k. \tag{I.22}$$

Taking covariant divergence of LHS and RHS of this equation we obtain

$$R^m_{i;\,m} - \frac{1}{2}\delta^m_i R_{;m} = \frac{8\pi G}{c^4} T^m_{k;\,m}, \tag{I.23}$$

hence

$$T^m_{k;\,m} = \frac{c^4}{8\pi G}\left(R^m_{i;\,m} - \frac{1}{2}\delta^m_i R_{;m}\right) = \frac{c^4}{8\pi G}\left(R^m_{i;\,m} - \frac{1}{2} R_{,i}\right) = 0. \tag{I.24}$$

b) Contracting the EFEs written in mixed form (we have

$$R^m_m - \frac{1}{2}\delta^m_m R = \frac{8\pi G}{c^4} T^m_m, \tag{I.25}$$

hence

$$R - \frac{4}{2} R = \frac{8\pi G}{c^4} T = \frac{8\pi G}{c^4}(\varepsilon - 3P), \text{ hence } R - 2R = \frac{8\pi G}{c^4}(\varepsilon - 3P), \text{ and finally, } R = -\frac{8\pi G}{c^4}(\varepsilon - 3P). \tag{I.26}$$

4. *Summary*

1) The Equivalence principle leads to the Principle of Covariance.
2) Gravity is manifestation of the geometry of space-time (Geometrical principle).
3) These principles predetermine the whole mathematical structure of GR.
4) To take into account any effect of gravity on any physical process it is enough to make replacement

$$d \to D, \quad , \to ;$$

5) EFEs describe the generation of gravity by matter with the arbitrary equation of state and we will use EFEs in Cosmology.

IV. SUPPLEMENTARY TO THE LECTURE 4

This is about General Relativistic cosmological models

A. Derivation of relativistic cosmological models

IV A 1 Geometry of isotropic and homogeneous Universe,
IV A 2 Three-dimensional space of constant curvature,
IV A 3 Friedman-Lemaî̂tre-Robertson-Walker (FLRW) metric,
IV A 4 Relativistic Acceleration Equation.

1. Geometry of isotropic and homogeneous Universe

What is the space-time geometry corresponding to isotropic and homogeneous Universe? Let us write the metric in the following form:

$$ds^2 = g_{ik}dx^i dx^k = g_{00}(dx^0)^2 + 2g_{0\alpha}dx^0 dx^\alpha - dl^2, \tag{A.1}$$

where

$$dl^2 = -g_{\alpha\beta}dx^\alpha dx^\beta. \tag{A.2}$$

We will work in the so called "co-moving" frame of reference, co-moving to the matter filling the Universe. The fact that all directions in a isotropic Universe are equivalent to each other, means that

$$g_{0\alpha} = 0, \tag{A.3}$$

otherwise $g_{0\alpha} \neq 0$ considered as 3-vector would lead to non-equivalence of different directions. The homogeneity of the Universe means that g_{00} can depend only on time coordinate, hence we can choose time coordinate t as

$$c^2dt^2 = g(x^0)_{00}(dx^0)^2, \tag{A.4}$$

to obtain

$$ds^2 = c^2dt^2 - dl^2. \tag{A.5}$$

2. Three-dimensional space of constant curvature

According to the cosmological principle the Universe is the same everywhere, as a consequence, The three-dimensional space is curved in the same way everywhere, which means that at each moment of time the metric of the space is the same at all points. To obtain such a metric let us start from the following geometrical analogy. Let us consider the two-dimensional sphere in the flat three-dimensional space. In this case the element of length is

$$dl^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2. \tag{A.6}$$

The equation of a sphere of radius a in the three-dimensional space has the form

$$(x^1)^2 + (x^2)^2 + (x^3)^2 = a^2. \tag{A.7}$$

The element of length on the two-dimensional sphere can be obtained if one expresses dx^3 in terms of dx^1 and dx^2. From the equation for sphere we have

$$x^1dx^1 + x^2dx^2 + x^3dx^3 = 0, \tag{A.8}$$

hence

$$dx^3 = -\frac{x^1dx^1 + x^2dx^2}{x^3} = -\frac{x^1dx^1 + x^2dx^2}{\sqrt{a^2 - (x^1)^2 - (x^2)^2}}. \tag{A.9}$$

Substituting (A.8) into (D.4) we have

$$dl^2 = (dx^1)^2 + (dx^2)^2 + \frac{(x^1dx^1 + x^2dx^2)^2}{a^2 - (x^1)^2 - (x^2)^2}. \tag{A.10}$$

Let us introduce the "polar" coordinates instead of x^1 and x^2

$$x^1 = r\cos\phi, \quad x^2 = r\sin\phi. \tag{A.11}$$

As a result we obtain

$$dl^2 = (dr\cos\phi - r\sin\phi d\phi)^2 + (dr\sin\phi + r\cos\phi d\phi)^2 +$$

$$\frac{r\cos\phi(dr\cos\phi - r\sin\phi d\phi) + r\sin\phi(dr\sin\phi + r\cos\phi d\phi)}{a^2 - r^2} = \frac{a^2dr^2}{a^2 - r^2} + r^2d\phi^2 = \frac{dr^2}{1 - \frac{r^2}{a^2}} + r^2d\phi^2. \tag{A.12}$$

Now we can repeat step by step the previous derivation, by considering the geometry of the three-dimensional space as the geometry on the three-dimensional hypersphere in some fictitious four-dimensional space (don't confuse with the physical four-dimensional space-time). In this case the element of length is

$$dl^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2. \tag{A.13}$$

The equation of a sphere of radius a in the four-dimensional space has the form

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = a^2. \tag{A.14}$$

The element of length on the three-dimensional hypersphere, which represents the three-dimensional space of constant curvature, can be obtained, if one expresses dx^4 in terms of dx^1, dx^2 and dx^3. From the equation for hypersphere we have

$$x^1dx^1 + x^2dx^2 + x^3dx^3 + x^4dx^4 = 0. \tag{A.15}$$

Hence

$$dx^4 = -\frac{x^1dx^1 + x^2dx^2 + x^3dx^3}{x^4} = -\frac{x^1dx^1 + x^2dx^2 + x^3dx^3}{\sqrt{a^2 - (x^1)^2 - (x^2)^2 - (x^3)^2}}. \tag{A.16}$$

Substituting (A.16) into (A.13) we have

$$dl^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + \frac{(x^1dx^1 + x^2dx^2 + x^3dx^3)^2}{a^2 - (x^1)^2 - (x^2)^2 - (x^3)^2}. \tag{A.17}$$

Let us now introduce the "spherical" coordinates instead of x^1, x^2 and x^3

$$x^1 = r\sin\theta\cos\phi, \quad x^2 = r\sin\theta\sin\phi, \quad x^3 = r\cos\theta. \tag{A.18}$$

As a result we obtain

$$dl^2 = (dr\sin\theta\cos\phi + r\cos\theta d\theta - r\sin\theta\sin\phi d\phi)^2 + (dr\sin\theta\sin\phi + r\cos\theta\cos\phi d\theta + r\sin\theta\sin\phi d\phi)^2 + (dr\cos\theta - r\sin\theta d\theta)^2 +$$

$$+\frac{1}{a^2 - r^2}[r\sin\theta\cos\phi(dr\sin\theta\cos\phi + r\cos\theta d\theta - r\sin\theta\sin\phi d\phi) + r\sin\theta\sin\phi(dr\sin\theta\sin\phi + r\cos\theta\cos\phi d\theta + r\sin\theta\sin\phi d\phi) +$$

$$+ r\cos\theta(dr\cos\theta - r\sin\theta d\theta)] = \frac{a^2dr^2}{a^2 - r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) = \frac{dr^2}{1 - \frac{r^2}{a^2}} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{A.19}$$

Taking into account that $r \leq a$ we can introduce instead of r the new lagrangian radial coordinate χ such that

$$r = a \sin\chi \ \text{ and } \ dr = a\cos\chi d\chi, \tag{A.20}$$

as a result dl can be rewritten as

$$dl^2 = a^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{A.21}$$

Now we can write the metric interval for the four dimensional space time as

$$ds^2 = c^2dt^2 - a^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{A.22}$$

We can repeat all these calculation for the three-dimensional space of the negative constant curvature. For that one should replace the equation (A.14) for the hypersphere by

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = -a^2, \tag{A.23}$$

which is a sphere of imaginary radius. This means that to obtain the metric of three-dimensional space of constant negative curvature one should just replace a by ia. Obviously, when $a \to \infty$ we obtain the case of the spatially flat space. This method is called the method of embedding diagrams. In the relativistic cosmology the curvature of the three-dimensional space is related with the density parameter.

3. *Friedman-Lemaître-Robertson-Walker (FLRW) metric*

The fact that the Universe is expanding means that instead of constant a we should introduce a scale factor $a(t)$ and finally we obtain the famous Friedmann-Lemaître-Robertson-Walker (FLRW) metric for expanding Universe

$$ds^2 = c^2dt^2 - a^2(t)[d\chi^2 + f^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)], \tag{A.24}$$

where

$$f = \left\{ \begin{array}{cc} \sin\chi & \text{for constant positive curvatue} \\ \sinh\chi & \text{for constant negative curvatue} \\ \chi & \text{for zero curvature} \end{array} \right\}. \tag{A.25}$$

The function $f(\chi)$ can be written in more elegant way as follows

$$f(\chi) = \frac{\sin A\chi}{A}, \tag{A.26}$$

where

$$A = \left\{ \begin{array}{cc} 1 & \text{for constant positive curvature} \\ i & \text{for constant negative curvature} \\ 0 & \text{for zero curvature} \end{array} \right\}. \tag{A.27}$$

Indeed, if $A = 1$, obviously $f(\chi) = \sin\chi$. If $A = i$

$$f(\chi) = \frac{\sin i\chi}{i} = \frac{e^{i\cdot iA} - e^{-i\cdot iA}}{2i\cdot i} = \frac{e^{-A} - e^{A}}{-2} = \frac{e^{A} - e^{-A}}{2} = \sinh\chi. \tag{A.28}$$

If $A = 0$

$$f(\chi) = \lim_{A\to 0}\frac{\sin A\chi}{A} = \chi. \tag{A.29}$$

Sometime it is convenient to introduce another time coordinate, η, called the conformal time and defined as

$$cdt = ad\eta. \tag{A.30}$$

In terms of this new time-coordinate (**??**) can be re-written as

$$ds^2 = a(\eta)[d\eta^2 - d\chi^2 - f^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{A.31}$$

Using the EFEs we can obtain required equations for $a(t)$, in other words we can obtain relativistic cosmological models based on the EFEs.

4. Relativistic Acceleration Equation

In order to avoid confusion between the scale factor a used in previous sections, with the scale curvature used in the present section, let us use a as the new notation for the scale factor. Contracting the EFEs written in mixed form we have

$$R^m_m - \frac{1}{2}\delta^m_m R = \frac{8\pi G}{c^4}T^m_m, \tag{A.32}$$

hence

$$R - \frac{4}{2}R = \frac{8\pi G}{c^4}T = \frac{8\pi G}{c^4}(\varepsilon - 3P), \quad R - 2R = \frac{8\pi G}{c^4}(\varepsilon - 3P), \quad R = -\frac{8\pi G}{c^4}(\varepsilon - 3P). \tag{A.33}$$

Then

$$R^i_k = \frac{1}{2}\delta^i_k R + \frac{8\pi G}{c^4}T^i_k = \frac{8\pi G}{c^4}T^i_k - \frac{1}{2}\delta^i_k\frac{8\pi G}{c^4}T = \frac{8\pi G}{c^4}(T^i_k - \frac{1}{2}\delta^i_k T). \tag{A.34}$$

$$R^0_0 = \frac{8\pi G}{c^4}(T^0_0 - \frac{1}{2}T) = \frac{8\pi G}{c^4}[\varepsilon - \frac{1}{2}(\varepsilon - 3P)] = \frac{4\pi G}{c^4}(\varepsilon + 3P) = \frac{4\pi G}{c^2}(\rho + \frac{3P}{c^2}). \tag{A.35}$$

$$R^0_0 = g^{0n}R_{n0} = R_{00} = \Gamma^l_{00,l} - \Gamma^l_{0l,0} + \Gamma^l_{00}\Gamma^m_{lm} - \Gamma^m_{0l}\Gamma^l_{0m}. \tag{A.36}$$

We can see that

$$\Gamma^l_{00} = \frac{g^{ln}}{2}\left(g_{0n,0} + g_{n0,0} - g_{00,n}\right) = g^{ln}\left(g_{0n,0} - \frac{1}{2}g_{00,n}\right) = g^{ln}g_{0n,0} = g^{l0}g_{00,0} + g^{l\alpha}g_{0\alpha,0} = 0. \tag{A.37}$$

Hence

$$R^0_0 = -\Gamma^l_{0l,0} - \Gamma^m_{0l}\Gamma^l_{0m} = -\Gamma^\alpha_{0\alpha,0} - \Gamma^\beta_{0\alpha}\Gamma^\alpha_{0\beta}. \tag{A.38}$$

Taking into account that

$$\Gamma^\alpha_{0\beta} = \frac{g^{\alpha n}}{2}\left(g_{0n,\beta} + g_{\beta n,0} - g_{0\beta,n}\right) = \frac{g^{\alpha n}}{2}g_{\beta n,0} = \frac{\dot{a}}{ca}\frac{g^{\alpha n}}{2}g_{\beta n} = \frac{\dot{a}}{ca}\delta^\alpha_\beta. \tag{A.39}$$

Thus

$$R^0_0 = -\left[\frac{d}{cdt}\left(\frac{\dot{a}}{ac}\right)\delta^\alpha_\alpha + \left(\frac{\dot{a}}{ac}\right)^2\delta^\beta_\alpha\delta^\alpha_\beta\right] = -\frac{3}{c^2}\left[\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 + \left(\frac{\dot{a}}{a}\right)^2\right] = -\frac{3\ddot{a}}{ac^2}. \tag{A.40}$$

Hence

$$-\frac{3\ddot{a}}{ac^2} = \frac{4\pi G}{c^2}(\rho + \frac{3P}{c^2}) \text{ and } \ddot{a} = -\frac{4\pi G}{3}(\rho + \frac{3P}{c^2})a. \tag{A.41}$$

This is the relativistic acceleration equation. It is interesting to compare Eq. (A.41) with the acceleration equation in the in the Newtonian theory:

$$\ddot{a_N} = -\frac{4\pi G\rho}{3}a_N. \tag{A.42}$$

One can see the great difference: $\ddot{a}$ is determined not only by density, as in Newtonian theory, but also by pressure (if $P > 0$) or tension (if $P < 0$).

B. Relativistic Friedman Equation (FE)

1. *The Energy Conservation Equation in GR*

The acceleration equation plus the equation of state contain three variables to find: the scale factor a, the energy density ε ($\varepsilon = \rho c^2$) and pressure P (when $P < 0$ it is better to consider P as a tension). Hence we need one more equation. Let us take as the third equation

$$T^i_{0;i} = 0, \tag{B.1}$$

which, as was shown in the previous lecture is the consequence of the EFEs and the Bianchi identity. Using the rules of the covariant differentiation we have

$$T^i_{0;i} = T^i_{0,i} + \Gamma^i_{in}T^n_0 - \Gamma^n_{i0}T^i_n = T^0_{0,0} + \Gamma^\alpha_{\alpha 0}T^0_0 - \Gamma^\alpha_{\beta 0}T^\beta_\alpha =$$

$$= \frac{1}{c}\dot{\epsilon} + \frac{\dot{a}}{ca}\delta^\alpha_\alpha \epsilon - \frac{\dot{a}}{ca}\delta^\alpha_\beta T^\beta_\alpha = \frac{1}{c}\left(\dot{\epsilon} + \frac{3\dot{a}}{a}\epsilon + \frac{\dot{a}}{a}\delta^\alpha_\beta \delta^\beta_\alpha P\right) = \frac{1}{c}\left[\dot{\epsilon} + \frac{3\dot{a}}{a}(\epsilon + P)\right] = 0. \tag{B.2}$$

Hence,

$$\dot{\rho} = -\frac{3\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right). \tag{B.3}$$

This is the energy conservation equation. If $P = 0$ this equation gives the equation of mass conservation for dust. It is interesting to mention that this equation is identical to the first law of thermodynamics which says that the change of energy ,$E = \rho c^2 V$, in volume, V, surrounded by the surface of area S is equal to the work done by the pressure forces, $F_P = SP$,

$$dE = -F_P dx = -PSdx = -PdV, \text{ or } \dot{E} = -P\dot{V}. \tag{B.4}$$

Let us consider a sphere of radius r, in this case

$$V = \frac{4\pi}{3}r^3, \text{ and } \left(r^3\rho\right)^{\cdot} = -\frac{P}{c^2}\left(r^3\right)^{\cdot}, \text{ then} \tag{B.5}$$

$$\dot{\rho}r^3 + 3r^2\dot{a}\rho = -\frac{P}{c^2}3r^2, \text{ i.e.} \dot{\rho} = -\frac{3\dot{r}}{r}\left(\rho + \frac{P}{c^2}\right). \tag{B.6}$$

Taking into account that in the homogeneously expanding Universe

$$\frac{\dot{r}}{r} = \frac{\dot{a}}{a}, \tag{B.7}$$

we obtain exactly the energy conservation equation derived from the EFEs and the pure geometrical Bianchi identity. It means that the first law of thermodynamics is a consequence of the EFEs! Actually we completed the required set of equations describing the relativistic cosmological models. Now we are going to derive from these two equations, the acceleration equation and the energy conservation equation, the relativistic version of the Friedman equation.

2. The FE for FLRW metric. Great surprise

From the energy conservation equation we obtain

$$\frac{P}{c^2} = -\rho - \frac{\dot{\rho}a}{3\dot{a}}. \tag{B.8}$$

Putting this expression for p into the acceleration equation, we have

$$\ddot{a} = -\frac{4\pi G a}{3}\left(\rho + \frac{3P}{c^2}\right) = -\frac{4\pi G a}{3}\left(\rho - 3\rho - \frac{\dot{\rho}a}{\dot{a}}\right) =$$

$$= \frac{4\pi G}{3\dot{a}}\left(2\rho a\dot{a} + \dot{\rho}a^2\right) = \frac{4\pi G}{3\dot{a}}\left(\rho a^2\right)^{\cdot}. \tag{B.9}$$

then multiplying both sides of this equation by $2\dot{a}$ and taking into account that

$$2\dot{a}\ddot{a} = \left(a^2\right)^{\cdot}, \tag{B.10}$$

we obtain

$$\left(\dot{a}^2\right)^{\cdot} = \frac{8\pi G}{3}\left(\rho a^2\right)^{\cdot}, \tag{B.11}$$

hence

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - kc^2. \tag{B.12}$$

This is the Friedmann equation in the relativistic Cosmology [k appears here as a dimensionless constant, exactly as in the Newtonian cosmological models, but as one can easily show it is related now with 3-curvature]. What a surprising result!: the Friedman equation is identical to the equation obtained in the Newtonian theory and does not contain the pressure term at all. However, the pressure is really extremely important. Indeed, if we put the equation of state

$$P = \alpha\rho c^2, \tag{B.13}$$

into the energy conservation equation we obtain

$$\frac{\dot{\rho}}{\rho} = -3(1+\alpha)\frac{\dot{a}}{a} \ \textbf{and} \ (\ln\rho)^{\cdot} + 3(1+\alpha)(\ln a)^{\cdot} = [\ln\rho + 3(1+\alpha)\ln a]^{\cdot} = \left\{\ln[\rho a^{3(1+\alpha)}]\right\}^{\cdot} = 0, \tag{B.14}$$

thus

$$\ln[\rho a^{3(1+\alpha)}] = C \ \text{ and } \ \rho a^{3(1+\alpha)} = C'. \tag{B.15}$$

Finally we obtain that

$$\rho = \rho_0\left(\frac{a_0}{a}\right)^{3(1+\alpha)}. \tag{B.16}$$

3. When the space curvature is negligible

If $k = 0$ the FE is reduced to

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3}. \tag{B.17}$$

Substituting into this equation the expression for ρ obtained above, we have

$$\frac{\dot{a}}{a} = \left(\frac{8\pi G\rho}{3}\right)^{1/2} \left(\frac{a_0}{a}\right)^{\frac{3(1+\alpha)}{2}}; \tag{B.18}$$

we can solve this equation by the separation of variables. For that let us introduce the following notations

$$x = \frac{a}{a_0}, \quad \beta = \frac{3(1+\alpha)}{2} \quad \text{and} \quad A = \left(\frac{8\pi G\rho}{3}\right)^{1/2}. \tag{B.19}$$

In terms of x, β and A the above equation can be written as

$$\dot{x}x^{\beta-1} = A, \;\; \text{hence,} \;\; \frac{1}{\beta}dx^{\beta} = Adt, \;\; \text{and} \;\; x^{\beta} = \beta At + C. \tag{B.20}$$

Since $x = 0$ at $t = 0$ we obtain that $C = 0$. Thus

$$x^{\beta} \sim t, \;\; \text{and} \;\; a \sim x \sim t^{\frac{1}{\beta}} = t^{\frac{2}{3(1+\alpha)}}. \tag{B.21}$$

In the case $k = \pm 1$ the solution (B.21) is still valid if

$$\frac{8\pi G\rho a^2}{3c^2} \gg |k| = 1. \tag{B.22}$$

As follows from Eqs.(B.16)the LHS of (B.22) goes as a^{β}, where $\beta = 2 - 3(1+\alpha) = -(1+3\alpha)$. Hence if $\alpha > -1/3$ the solution (B.21) is valid for small a, if $\alpha < -1/3$ (B.21)is valid for large a. WE will see during the next lecture that the latter case corresponds to the expansion with acceleration.

C. Expansion of the Universe in dependence of its content

1. *Inflation and expansion with acceleration*

Let the scale factor depends on time as

$$a \sim t^{\gamma}. \tag{C.1}$$

Taking into account that the physical distances between any two remote objects in the Universe are proportional to a, while the cosmological horizon is proportional to t, it is clear that if $\gamma > 1$ sooner or later any two objects will be outside the cosmological horizon, i.e. will be causally disconnected. Such fast expansion of the Universe is called inflation. On other hand, the condition that the Universe is expanding with acceleration rather than with deceleration,

$$\ddot{a} \sim \gamma(\gamma - 1)t^{\gamma-2} > 0, \tag{C.2}$$

means that if $\gamma > 1$ the universe expands with acceleration. In order words we can say that the inflation always implies the expansion with acceleration. From the previous section we know that γ is related to the equation of state parameter α as

$$\gamma = \frac{2}{3(1+\alpha)}. \tag{C.3}$$

Thus, the condition of inflation can be written as

$$\frac{2}{3(1+\alpha)} > 1, \quad \alpha = 1-\gamma < -\frac{1}{3}. \tag{C.4}$$

If $\alpha = -1/3$ and $\gamma = 1$ the acceleration od expansion is equal to zero. Such situation corresponds to domination of cosmic strings. If $\alpha = -1$ corresponds to domination of dark energy in the form of Λ-term. Indeed, in this case, the stress-energy tensor is

$$T_{ik} = (\epsilon + P)u_i u_k - g_{ik}P = \epsilon g_{ik} \tag{C.5}$$

and as follows from the conservation of energy equation

$$\dot{\epsilon} = -\frac{3\dot{a}}{a}(\epsilon + P) = 0, \tag{C.6}$$

which means that $\epsilon =$ const, hence T_{ik} has the same form as the Λ-term, i.e. const g_{ik}. In this case $\gamma \to \infty$ and formally we can not apply the power low to the evolution of the scalar parameter and should consider this case separately. As follows from the Friedman equation for spatially flat Universe ($k = 0$) the Hubble constant

$$H = \frac{\dot{a}}{a} = \sqrt{\frac{8\pi G\epsilon}{3c^2}} = \text{const}. \tag{C.7}$$

Thus the scale factor satisfies the following equation

$$\dot{a} - Ha = 0 \tag{C.8}$$

and as we already know from Lecture 12, the obvious solution of this equation is

$$a \sim e^{Ht}. \tag{C.9}$$

This is the exponential inflation.

2. *The Hubble constant as a function of the scale factor*

The real Universe contains different forms of matter, radiation and dark energy. Let us call them components. At different epochs the different components dominate. Let us consider the situation when there are several not interacting with each other components. Let ascribe to each component some number $i = 1, 2, 3...N$. Each component is characterized by its own equation of sate

$$P_{(i)} = \alpha_{(i)}\rho_{(i)}c^2. \tag{C.10}$$

The density of each component evolves according to its own law

$$\rho_{(i)}(t) = \rho_{(i)}(t_0)\left(\frac{a}{a_0}\right)^{-3(1+\alpha_{(i)})} = \rho_{cr}\Omega_{(i)}\left(\frac{a}{a_0}\right)^{-3(1+\alpha_{(i)})}, \quad \text{where} \quad \rho_{cr} = \frac{3H_0^2}{8\pi G} \tag{C.11}$$

is the critical density at the present moment and $\Omega_{(i)}$ is the dimensionless density parameter taken also at the present moment. According to the Friedman equation for spatially flat universe

$$H^2 = \frac{8\pi G\rho}{3}, \quad \text{where} \quad \rho = \Sigma_{i=1}^N \rho_{(i)}(t) = \rho_{cr}\Sigma_{i=1}^N \Omega_{(i)}\left(\frac{a}{a_0}\right)^{-3(1+\alpha_{(i)})}, \quad \text{hence} \quad H = H_0\sqrt{\Sigma_{i=1}^N \Omega_{(i)}\left(\frac{a}{a_0}\right)^{-3(1+\alpha_{(i)})}} \tag{C.12}$$

D. Examples and Problems

1. The FLRW metric

Problem: Starting from the metric

$$ds^2 = c^2dt^2 - \frac{dr^2}{1 - \frac{kr^2}{a^2}} - r^2(d\theta^2 + \sin^2\theta d\phi^2, \tag{D.1}$$

where $k = 0, 1$ or -1, and assuming that a is a function of t only, show that
a) this metric is the FLRW metric and can be presented as

$$ds^2 = c^2dt^2 - a(t)^2[d\chi^2 + \frac{\sin^2\sqrt{k}\chi}{k}(d\theta^2 + \sin^2\theta d\phi^2)]; \tag{D.2}$$

b) this metric can be presented as

$$ds^2 = c^2dt^2 - a(t)^2[\frac{d\sigma^2}{1 - k\sigma^2} + \sigma^2(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{D.3}$$

c) Find the relationship between χ and σ.
Solution:
a) Let us consider first

$$dl^2 = \frac{dr^2}{1 - \frac{kr^2}{a^2}} + r^2d\Omega^2, \tag{D.4}$$

where a is a constant and

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2. \tag{D.5}$$

Let us introduce a radial lagrangian coordinate χ as

$$r = \frac{a}{\sqrt{k}}\sin\sqrt{k}\chi. \tag{D.6}$$

Then let us substitute (D.6) into (D.4). As a result we obtain

$$dl^2 = \frac{(\sqrt{k}a\cos\sqrt{k}\chi\frac{1}{\sqrt{k}}d\chi)^2}{1 - \sin^2\sqrt{k}\chi} + \frac{a^2}{k}\sin^2\sqrt{k}\chi d\Omega^2 = a^2\left[\frac{\cos^2\sqrt{k}\chi}{1 - \sin^2\sqrt{k}\chi}d\chi^2 + \frac{\sin^2\sqrt{k}\chi}{k}d\Omega^2\right]. \tag{D.7}$$

Then treating the radius of curvature as a function of time, $a = a(t)$ and using

$$ds^2 = c^2dt^2 - dl^2, \text{ we obtain the required metric.} \tag{D.8}$$

b) Let us introduce another radial lagrangian coordinate σ as

$$r = a\sigma. \tag{D.9}$$

Let us then substitute (D.9) into (A.13), we obtain

$$dl^2 = \frac{(ad\sigma)^2}{1-k\sigma^2} + a^2\sigma^2 d\Omega^2 = a^2\left[\frac{d\sigma^2}{1-k\sigma^2} + \sigma^2 d\Omega^2\right]. \tag{D.10}$$

Then treating the radius of curvature as a function of time, $a = a(t)$ and using

$$ds^2 = c^2dt^2 - dl^2, \tag{D.11}$$

we obtain the required metric.
c) Taking into account that

$$r = \frac{a}{\sqrt{k}} \sin\sqrt{k}\chi = a\sigma, \tag{D.12}$$

we obtain

$$\sigma = \frac{\sin\sqrt{k}\chi}{\sqrt{k}}. \tag{D.13}$$

2. The equation of state and the early Universe.

Problem: A cosmological model describes the early Universe which contains a perfect fluid with equation of state $P = \alpha\rho c^2$. Using the energy conservation and acceleration equations (see rubric) express acceleration parameter q in terms of α.
Solution: From conservation of energy we have

$$d(\rho c^2 a^3) = -\alpha\rho c^2 d(a^3), \tag{D.14}$$

hence

$$\frac{d\rho(a)}{\rho} = -3(1+\alpha)\frac{da}{a} \tag{D.15}$$

and

$$\rho \propto a^{-3(1+\alpha)}. \tag{D.16}$$

The fact that we consider the early Universe means that we can neglect the curvature term in the Friedman equation. Thus from the Friedman equation we have

$$\dot{a} = \sqrt{\frac{8\pi G\rho}{3}}a \propto a^{1-\frac{3}{2}(1+\alpha)} = a^{-\frac{3\alpha+1}{2}}. \tag{D.17}$$

This means that

$$dt \propto a^{\frac{3\alpha+1}{2}}da \text{ i.e. } t \propto a^{\frac{3\alpha+1}{2}+1} = a^{\frac{3(1+\alpha)}{2}}, \text{ hence } a \propto t^{\frac{2}{3(1+\alpha)}} \text{ and } a = a_0\left(\frac{t}{t_0}\right)^{\frac{2}{3(1+\alpha)}}. \tag{D.18}$$

Then the deceleration parameter is

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = -\frac{2}{3(1+\alpha)}\left(\frac{2}{3(1+\alpha)} - 1\right)\left(\frac{2}{3(1+\alpha)}\right)^{-2} = \frac{3(1+\alpha)}{2} - 1. \tag{D.19}$$

3. Dark energy

Problem: Consider a spatially flat cosmological model containing dark energy with equation of state $\alpha = -2/3$ and dust with density parameter $\Omega_{0(d)}$. Find $\Omega_{0(d)}$ if it is given that according to a such model the Universe started expand with acceleration at $a = 2a_0/3$, where a_0 is the current scale factor.
Solution: The fact that the dark energy and dust do not interact with each other means that we can write down the conservation of energy equation for dark energy and dust separately. For dust we have

$$\rho_{(d)} \propto a^{-3}. \tag{D.20}$$

For dark energy we have

$$\rho_{(de)} = \rho_{0(de)} \left(\frac{a}{a_0}\right)^{-3(1-2/3)} = \Omega_{0(de)}\rho_{crit} \left(\frac{a}{a_0}\right)^{-1} \tag{D.21}$$

and

$$P_{(de)} = \alpha_{(de)} c^2 \rho_{(de)} = -\frac{2\Omega_{0(de)\rho_{crit}}}{3} \left(\frac{a}{a_0}\right)^{-1}. \tag{D.22}$$

We know that

$$\Omega_0 = \Omega_{0(d)} + \Omega_{0(de)} = 1, \quad \rho = \rho_{(d)} + \rho_{(de)}, \quad P = P_{(de)}, \tag{D.23}$$

we finally obtain the required formulae for $P/\rho c^2$. The Universe starts to expand with acceleration at the moment when the trace of stress-energy tensor, $\rho c^2 + 3P$, changes sign. From the previous eqs. we have

$$\left[\Omega_{0(d)} \left(\frac{a}{a_0}\right)^{-3} + (1 - \Omega_{0(d)}) \left(\frac{a}{a_0}\right)^{-1} - 2(1 - \Omega_{0(r)}) \left(\frac{a}{a_0}\right)^{-1}\right] = 0. \tag{D.24}$$

Hence

$$\frac{9}{4}\Omega_{0(d)} + (1 - \Omega_{0(d)})(1 - 2) = 0, \tag{D.25}$$

finally

$$\Omega_{0(d)} = \frac{9}{13}. \tag{D.26}$$

4. Λ-term

Problem: There is the following version of the EFEs.

$$R^i_k - \frac{1}{2}\delta^i_k R - \Lambda\delta^i_k = \frac{8\pi G}{c^4} T^i_k, \tag{D.27}$$

where T^i_k is the Stress-Energy tensor and Λ is a constant introduced by Einstein.
a)Show that in this case the acceleration equation and the Friedman equation can be written as

$$\ddot{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) a + \frac{\Lambda a}{3}, \tag{D.28}$$

$$\dot{a}^2 + kc^2 = \frac{8\pi G}{3}\rho a^2 + \frac{\Lambda a^2}{3} \tag{D.29}$$

b) Show that a positive Λ - term corresponds to a repulsive force whose strength is proportional to distance. Show that when Λ - term dominates

$$a \sim exp[(\Lambda/3)^{1/2} t]. \tag{D.30}$$

c) Show that Λ-term corresponds to dark energy with $\alpha = 1$.

V. SUPPLEMENTARY TO THE LECTURE 5

VI. SUPPLEMENTARY TO THE LECTURE 6

VII. SUPPLEMENTARY TO THE LECTURE 7

VIII. SUPPLEMENTARY TO THE LECTURE 8

IX. SUPPLEMENTARY TO THE LECTURE 9

X. SUPPLEMENTARY TO THE LECTURE 10

XI. SUPPLEMENTARY TO THE LECTURE 11