Part-III Cosmology

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Part III — Large-scale structure formation

The real universe is far from homogeneous and isotropic. Figure 1 shows slices through the 3D distribution of galaxy positions from the 2dF galaxy redshift survey out to a comoving distance of 600 Mpc. The distribution of galaxies is clearly not random; instead they are arranged into a delicate *cosmic web* with galaxies strung out along dense filaments and clustering at their intersections leaving huge empty voids. However, if we smooth the picture on large scales ($\sim$ 100 Mpc) is starts to look much more homogeneous. Furthermore, we know from the CMB that the universe was smooth to around 1 part in 10^5 at the time of recombination; see Fig. 2. The aim of this part of the course is to study the growth of large-scale structure in an expanding universe through *gravitational instability* acting on small initial perturbations. We shall then learn how these initial perturbations were likely produced by quantum effects during cosmological inflation.

This final part of the course is structured as follows:

- Statistics of random fields
- Newtonian structure formation
- Introduction to relativistic perturbation theory
- CMB temperature anisotropies
- Fluctuations from inflation

3.1 Statistics of random fields

Theory (e.g. quantum mechanics during inflation) only allows us to predict the statistical properties of cosmological fields (such as the matter overdensity $\delta\rho$). Here, we explore the basic statistical properties enforced on such fields by assuming the physics that generates the initial fluctuations, and subsequently processes them, respects the symmetries of the background cosmology, i.e. isotropy and homogeneity.

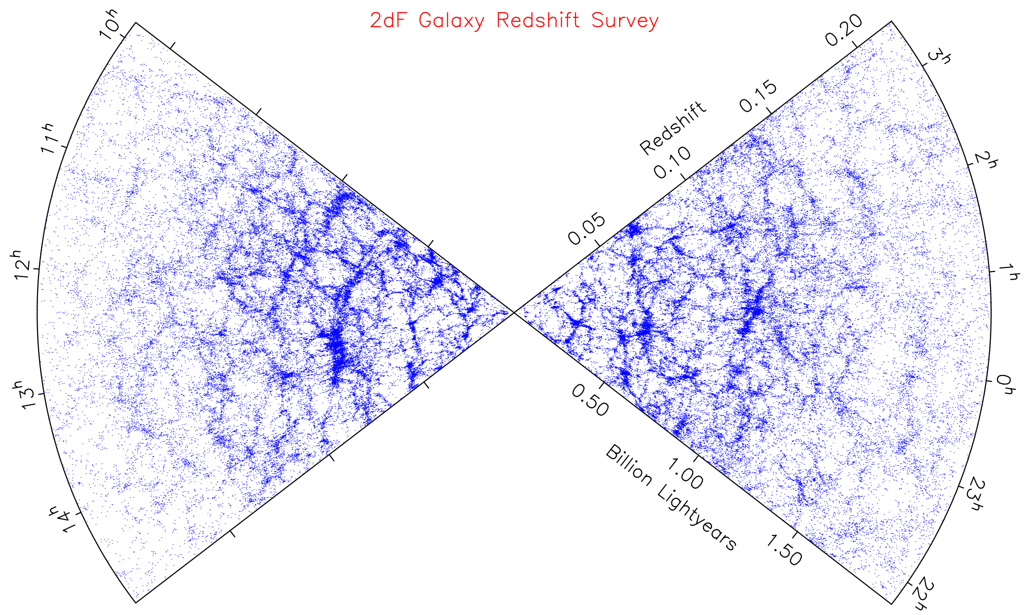

Figure 1: Slices through the 3D map of galaxy positions from the 2dF galaxy redshift survey. Note that redshift 0.15 is at a comoving distance of 600 Mpc. Credit: 2dF.

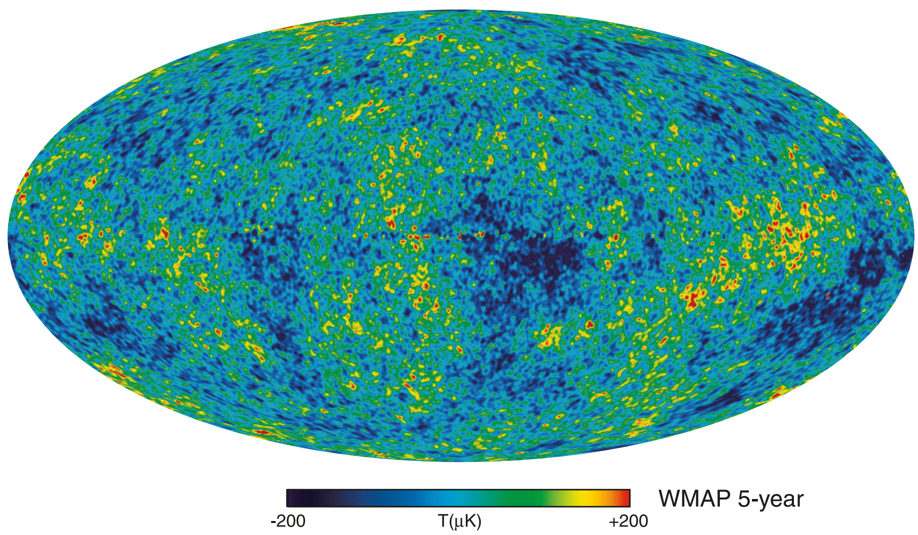

Figure 2: Fluctuations in the CMB temperature, as determined from five years of WMAP data, about the average temperature of 2.725 K. The fluctuations are at the level of only a few parts in 10^5. Credit: WMAP science team.

Throughout, we denote expectation values with angle brackets, e.g. $\langle \delta\rho \rangle$; you should think of this as a quantum expectation value or an average over a classical ensemble[1].

[1]For a recent review on the question of why quantum fluctuations from inflation can be treated as

To keep the Fourier analysis simple, we shall only consider flat ($K = 0$) background models and we denote comoving spatial positions by $\mathbf{x}$.

3.1.1 Random fields in 3D Euclidean space

Consider a random field $f(\mathbf{x})$ – i.e. at each point $f(\mathbf{x})$ is some random number – with zero mean, $\langle f(\mathbf{x})\rangle = 0$. The probability of realising some field configuration is a *functional* $\Pr[f(\mathbf{x})]$. *Correlators* of fields are expectation values of products of fields at different spatial points (and, generally, times). The two point correlator is

$$\xi(\mathbf{x},\mathbf{y}) \equiv \langle f(\mathbf{x})f(\mathbf{y})\rangle = \int \mathcal{D}f \,\Pr[f] f(\mathbf{x})f(\mathbf{y})\,, \tag{3.1.1}$$

where the integral is a *functional integral* (or path integral) over field configurations.

Statistical homogeneity means that the statistical properties of the translated field,

$$\hat{T}_{\mathbf{a}} f(\mathbf{x}) \equiv f(\mathbf{x}-\mathbf{a})\,, \tag{3.1.2}$$

are the same as the original field, i.e. $\Pr[f(\mathbf{x})] = \Pr[\hat{T}_{\mathbf{a}} f(\mathbf{x})]$. For the two-point correlation, this means that

$$\begin{aligned} &\xi(\mathbf{x},\mathbf{y}) = \xi(\mathbf{x}-\mathbf{a},\mathbf{y}-\mathbf{a}) \quad \forall\mathbf{a} \\ \Rightarrow \quad &\xi(\mathbf{x},\mathbf{y}) = \xi(\mathbf{x}-\mathbf{y})\,, \end{aligned} \tag{3.1.3}$$

so the two-point correlator only depends on the separation of the two points.

Statistical isotropy mean that the statistical properties of the rotated field,

$$\hat{R} f(\mathbf{x}) \equiv f(\mathsf{R}^{-1}\mathbf{x})\,, \tag{3.1.4}$$

where R is a rotation matrix, are the same as the original field, i.e. $\Pr[f(\mathbf{x})] = \Pr[\hat{R} f(\mathbf{x})]$. For the two-point correlator, we must have

$$\xi(\mathbf{x},\mathbf{y}) = \xi(\mathsf{R}^{-1}\mathbf{x},\mathsf{R}^{-1}\mathbf{y}) \quad \forall\mathsf{R}\,. \tag{3.1.5}$$

Combining statistical homogeneity and isotropy gives

$$\begin{aligned} &\xi(\mathbf{x},\mathbf{y}) = \xi\left(\mathsf{R}^{-1}(\mathbf{x}-\mathbf{y})\right) \quad \forall\mathsf{R} \\ \Rightarrow \quad &\xi(\mathbf{x},\mathbf{y}) = \xi(|\mathbf{x}-\mathbf{y}|)\,, \end{aligned} \tag{3.1.6}$$

so the two-point correlator depends only on the distance between the two points. Note that this holds even if correlating fields at different times, or correlating different fields.

classical, see Keifer & Polarski (2008), available online at http://arxiv.org/abs/0810.0087.

We can repeat these arguments to constrain the form of the correlators in Fourier space. We adopt the symmetric Fourier convention, so that

$$f(\mathbf{k}) = \int \frac{d^3\mathbf{x}}{(2\pi)^{3/2}} f(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \quad \text{and} \quad f(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} f(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} . \tag{3.1.7}$$

Note that for real fields, $f(\mathbf{k}) = f^*(-\mathbf{k})$. Under translations, the Fourier transform acquires a phase factor:

$$\begin{aligned} \hat{T}_{\mathbf{a}} f(\mathbf{k}) &= \int \frac{d^3\mathbf{x}}{(2\pi)^{3/2}} f(\mathbf{x}-\mathbf{a}) e^{-i\mathbf{k}\cdot\mathbf{x}} \\ &= \int \frac{d^3\mathbf{x}'}{(2\pi)^{3/2}} f(\mathbf{x}') e^{-i\mathbf{k}\cdot\mathbf{x}'} e^{-i\mathbf{k}\cdot\mathbf{a}} \\ &= f(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{a}} . \end{aligned} \tag{3.1.8}$$

Invariance of the two-point correlator in Fourier space is then

$$\begin{aligned} &\langle f(\mathbf{k}) f^*(\mathbf{k}')\rangle = \langle f(\mathbf{k}) f^*(\mathbf{k}')\rangle e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{a}} \quad \forall \mathbf{a} \\ \Rightarrow \quad &\langle f(\mathbf{k}) f^*(\mathbf{k}')\rangle = F(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}') , \end{aligned} \tag{3.1.9}$$

for some (real) function $F(\mathbf{k})$. We see that different Fourier modes are *uncorrelated*. Under rotations,

$$\begin{aligned} \hat{R} f(\mathbf{k}) &= \int \frac{d^3\mathbf{x}}{(2\pi)^{3/2}} f(\mathsf{R}^{-1}\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \\ &= \int \frac{d^3\mathbf{x}}{(2\pi)^{3/2}} f(\mathsf{R}^{-1}\mathbf{x}) e^{-i(\mathsf{R}^{-1}\mathbf{k})\cdot(\mathsf{R}^{-1}\mathbf{x})} \\ &= f(\mathsf{R}^{-1}\mathbf{k}) , \end{aligned} \tag{3.1.10}$$

so, additionally demanding invariance of the two-point correlator under rotations implies

$$\langle \hat{R} f(\mathbf{k}) [\hat{R} f(\mathbf{k}')]^*\rangle = \langle f(\mathsf{R}^{-1}\mathbf{k}) f^*(\mathsf{R}^{-1}\mathbf{k}')\rangle = F(\mathsf{R}^{-1}\mathbf{k})\delta(\mathbf{k}-\mathbf{k}') = F(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}') \quad \forall \mathsf{R} . \tag{3.1.11}$$

(We have used $\delta(\mathsf{R}^{-1}\mathbf{k}) = \det\mathsf{R}\,\delta(\mathbf{k}) = \delta(\mathbf{k})$ here.) This is only possible if $F(\mathbf{k}) = F(k)$ where $k \equiv |\mathbf{k}|$. We can therefore define the *power spectrum*, $\mathcal{P}_f(k)$, of a homogeneous and isotropic field, $f(\mathbf{x})$, by

$$\langle f(\mathbf{k}) f^*(\mathbf{k}')\rangle = \frac{2\pi^2}{k^3} \mathcal{P}_f(k) \delta(\mathbf{k}-\mathbf{k}') . \tag{3.1.12}$$

The normalisation factor $2\pi^2/k^3$ in the definition of the power spectrum is conventional and has the virtue of making $\mathcal{P}_f(k)$ dimensionless if $f(\mathbf{x})$ is.

The correlation function is the Fourier transform of the power spectrum:

$$\begin{aligned}\langle f(\mathbf{x})f(\mathbf{y})\rangle &= \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\frac{d^3\mathbf{k}'}{(2\pi)^{3/2}} \underbrace{\langle f(\mathbf{k})f^*(\mathbf{k}')\rangle}_{\frac{2\pi^2}{k^3}\mathcal{P}_f(k)\delta(\mathbf{k}-\mathbf{k}')} e^{i\mathbf{k}\cdot\mathbf{x}}e^{-i\mathbf{k}'\cdot\mathbf{y}} \\ &= \frac{1}{4\pi}\int \frac{dk}{k}\mathcal{P}_f(k)\int d\Omega_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} . \end{aligned} \tag{3.1.13}$$

We can evaluate the angular integral by taking $\mathbf{x}-\mathbf{y}$ along the z-axis in Fourier space. Setting $\mathbf{k}\cdot(\mathbf{x}-\mathbf{y}) = k|\mathbf{x}-\mathbf{y}|\mu$, the integral reduces to

$$2\pi\int_{-1}^{1} d\mu\, e^{ik|\mathbf{x}-\mathbf{y}|\mu} = 4\pi j_0(k|\mathbf{x}-\mathbf{y}|)\,, \tag{3.1.14}$$

where $j_0(x) = \sin(x)/x$ is a spherical Bessel function of order zero. It follows that

$$\xi(\mathbf{x},\mathbf{y}) = \int \frac{dk}{k}\,\mathcal{P}_f(k) j_0(k|\mathbf{x}-\mathbf{y}|)\,. \tag{3.1.15}$$

Note that this only depends on $|\mathbf{x}-\mathbf{y}|$ as required by Eq. (3.1.6).

The variance of the field is $\xi(0) = \int dk\,\mathcal{P}_f(k)/k$. A *scale-invariant* spectrum has $\mathcal{P}(k) = \text{const.}$ and its variance receives equal contributions from every decade in k.

3.1.2 Gaussian random fields

For a Gaussian (homogeneous and isotropic) random field, $\Pr[f(\mathbf{x})]$ is a Gaussian functional of $f(\mathbf{x})$. If we think of discretising the field in N pixels, so it is represented by a N-dimensional vector $\mathbf{f} = [f(\mathbf{x}_1), f(\mathbf{x}_2), \ldots, f(\mathbf{x}_N)]^T$, the probability density function for $\mathbf{f}$ is a multi-variate Gaussian fully specified by the correlation function

$$\langle f_i f_j\rangle = \xi(|\mathbf{x}_i - \mathbf{x}_j|) \equiv \xi_{ij}\,, \tag{3.1.16}$$

where $f_i \equiv f(\mathbf{x}_i)$, so that

$$\Pr(\mathbf{f}) \propto \frac{e^{-f_i\xi_{ij}^{-1}f_j}}{\sqrt{\det(\xi_{ij})}}\,. \tag{3.1.17}$$

Since $f(\mathbf{k})$ is linear in $f(\mathbf{x})$, the probability distribution for $f(\mathbf{k})$ is also a multi-variate Gaussian. Since different Fourier modes are uncorrelated (see Eq. 3.1.9), they are statistically *independent* for Gaussian fields.

Inflation predicts fluctuations that are very nearly Gaussian and this property is preserved by *linear* evolution. The cosmic microwave background probes fluctuations mostly in the linear regime and so the fluctuations look very Gaussian (see Fig. 2). Non-linear structure formation at late times destroys Gaussianity and gives the filamentary cosmic web (see Fig. 1). Searching for primordial non-Gaussianity to probe departures from simple inflation is a very hot topic but no convincing evidence for primordial non-Gaussianity has yet been found.

3.1.3 Random fields on the sphere

Spherical harmonics form a basis for (square-integrable) functions on the sphere:

$$f(\hat{\mathbf{n}}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{lm} Y_{lm}(\hat{\mathbf{n}}) \,. \tag{3.1.18}$$

The Y_{lm} are familiar from quantum mechanics as the position-space representation of the eigenstates of $\hat{L}^2 = -\nabla^2$ and $\hat{L}_z = -i\partial_\phi$:

$$\begin{aligned} \nabla^2 Y_{lm} &= -l(l+1) Y_{lm} \\ \partial_\phi Y_{lm} &= im Y_{lm} \,, \end{aligned} \tag{3.1.19}$$

with l an integer ≥ 0 and m an integer with $|m| \leq l$. The spherical harmonics are orthonormal over the sphere,

$$\int d\hat{\mathbf{n}}\, Y_{lm}(\hat{\mathbf{n}}) Y^*_{l'm'}(\hat{\mathbf{n}}) = \delta_{ll'}\delta_{mm'} \,. \tag{3.1.20}$$

There are various phase conventions for the Y_{lm}; here we adopt $Y^*_{lm} = (-1)^m Y_{l\,-m}$ so that $f^*_{lm} = (-1)^m f_{l\,-m}$ for a real field.

What is the implication of statistical isotropy for the correlators of f_{lm}? For the two-point correlator, it turns out that we must have[2]

$$\langle f_{lm} f^*_{l'm'} \rangle = C_l \delta_{ll'} \delta_{mm'} \,, \tag{3.1.21}$$

where C_l is the *angular power spectrum* of f. What does this imply for the two-point correlation function? We have

$$\begin{aligned} \langle f(\hat{\mathbf{n}}) f(\hat{\mathbf{n}}') \rangle &= \sum_{lm} \sum_{l'm'} \underbrace{\langle f_{lm} f^*_{l'm'} \rangle}_{C_l \delta_{ll'} \delta_{mm'}} Y_{lm}(\hat{\mathbf{n}}) Y^*_{l'm'}(\hat{\mathbf{n}}') \\ &= \sum_l C_l \underbrace{\sum_m Y_{lm}(\hat{\mathbf{n}}) Y^*_{lm}(\hat{\mathbf{n}}')}_{\frac{2l+1}{4\pi} P_l(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}')} = C(\theta) \,, \end{aligned} \tag{3.1.22}$$

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[2] A plausibility argument is as follows. Under rotations, the subset of the Y_{lm} with a given l (so $2l+1$ elements) transforms irreducibly so the $\delta_{ll'}$ form of the correlator is preserved under rotation. For rotation through γ about the z-axis,

$$Y_{lm}(\theta,\phi) \to Y_{lm}(\theta, \phi - \gamma) = e^{-im\gamma} Y_{lm}(\theta,\phi) \qquad \Rightarrow \qquad f_{lm} \to e^{-im\gamma} f_{lm} \,.$$

Under rotations,

$$\langle f_{lm} f^*_{l'm'} \rangle \to e^{-im\gamma} e^{im'\gamma} \langle f_{lm} f^*_{l'm'} \rangle \,,$$

so invariance requires the correlator be $\propto \delta_{mm'}$.

where $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}' = \cos\theta$ and we used the addition theorem for spherical harmonics to express the sum of products of the Y_{lm} in terms of the Legendre polynomials $P_l(x)$. It follows that the two-point correlation function depends only on the angle between the two points, as required by statistical isotropy. Note that the variance of the field is $\sum_l (2l+1)C_l/4\pi$.

We can invert the correlation function to get the power spectrum by using orthogonality of the Legendre polynomials:

$$C_l = 2\pi \int_{-1}^{1} d\cos\theta\, C(\theta) P_l(\cos\theta)\,. \tag{3.1.23}$$

3.2 Newtonian structure formation

Newtonian gravity is an adequate approximation of general relativity in cosmology on scales well inside the Hubble radius and when describing non-relativistic matter (for which the pressure P is much less than the energy density ρ). Newtonian gravity underlies all cosmological N-body simulations of the non-linear growth of structure and is much more intuitive than the full linearised treatment of general relativity (to come later, and, in more detail, next term in *Advanced Cosmology*). In particular, in cosmology we can use the Newtonian treatment to describe sub-Hubble fluctuations in the cold dark matter (CDM) and baryons after decoupling.

Consider an ideal, self-gravitating non-relativistic fluid with density (for this section only, the *mass* density which, given our assumptions is essentially the total energy density) ρ, pressure $P \ll \rho$ and velocity $\mathbf{u}$. Denote the usual Newtonian position vector by $\mathbf{r}$ and time by t. The equations of motion of the fluid are as follows:

$$\text{Continuity} \qquad \partial_t \rho + \boldsymbol{\nabla}_{\mathbf{r}} \cdot (\rho \mathbf{u}) = 0 \tag{3.2.1}$$

$$\text{Euler} \qquad \partial_t \mathbf{u} + \mathbf{u} \cdot \boldsymbol{\nabla}_{\mathbf{r}} \mathbf{u} = -\frac{1}{\rho} \boldsymbol{\nabla}_{\mathbf{r}} P - \boldsymbol{\nabla}_{\mathbf{r}} \Phi \tag{3.2.2}$$

$$\text{Poisson} \qquad \boldsymbol{\nabla}_{\mathbf{r}}^2 \Phi = 4\pi G \rho\,, \tag{3.2.3}$$

where the gravitational potential Φ determines the gravitational acceleration by $\mathbf{g} = -\boldsymbol{\nabla}_{\mathbf{r}} \Phi$. We can fudge the Poisson equation to get the correct Friedmann equations (see later) including the cosmological constant Λ by taking

$$\boldsymbol{\nabla}_{\mathbf{r}}^2 \Phi = 4\pi G \rho - \Lambda\,. \tag{3.2.4}$$

3.2.1 Background cosmology

To recover the background dynamics (described by the Friedmann equations), consider a uniform expanding ball of fluid satisfying Hubble's law $\mathbf{u} = H(t)\mathbf{r}$. (Note the velocity goes to the speed of light at the Hubble radius!) With $\Phi = 0$ at $\mathbf{r} = 0$, the Poisson equation (3.2.4) integrates as

$$\begin{aligned} \frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) &= (4\pi G\rho - \Lambda)r^2 \\ \Rightarrow \qquad \frac{\partial\Phi}{\partial r} &= \frac{1}{3}(4\pi G\rho - \Lambda)r \\ \Rightarrow \qquad \Phi &= \frac{1}{6}(4\pi G\rho - \Lambda)r^2 \,. \end{aligned} \tag{3.2.5}$$

The Euler equation then becomes

$$\begin{aligned} \frac{\partial H}{\partial t}\mathbf{r} + H^2\underbrace{\mathbf{r}\cdot\boldsymbol{\nabla}_{\mathbf{r}}\mathbf{r}}_{\mathbf{r}} &= -\frac{1}{3}(4\pi G\rho - \Lambda)\mathbf{r} \\ \Rightarrow \qquad \frac{\partial H}{\partial t} + H^2 &= \frac{1}{3}(\Lambda - 4\pi G\rho)\,. \end{aligned} \tag{3.2.6}$$

This is the Newtonian limit of one of the Friedmann equations (the relativistic result replaces ρ with the sum of the energy density and three times the pressure, $\rho + 3P$).

The continuity equation becomes

$$\begin{aligned} \partial_t\rho + \boldsymbol{\nabla}_{\mathbf{r}}\cdot[\rho(t)H(t)\mathbf{r}] &= 0 \\ \Rightarrow \qquad \partial_t\rho + 3\rho H &= 0\,. \end{aligned} \tag{3.2.7}$$

This is the usual Friedmann statement of energy conservation for $\rho \ll P$. Introducing the scale factor a via $\partial_t a/a = H$, we have

$$\frac{1}{\rho}\frac{\partial\rho}{\partial t} + 3\frac{\partial a}{\partial t} = 0 \quad \Rightarrow \quad \rho \propto a^{-3}\,, \tag{3.2.8}$$

which describes the dilution of the mass density by expansion.

Equations (3.2.6) and (3.2.7) have a first integral

$$K = a^2\left(H^2 - \frac{8\pi G}{3}\rho - \frac{1}{3}\Lambda\right)\,. \tag{3.2.9}$$

This is easily checked by differentiating:

$$\begin{aligned} \frac{\partial K}{\partial t} &= 2a^2H\left(H^2 - \frac{8\pi G}{3}\rho - \frac{1}{3}\Lambda\right) + a^2\left(2H\frac{\partial H}{\partial t} - \frac{8\pi G}{3}\frac{\partial\rho}{\partial t}\right) \\ &= a^2\left[2H^3 - \frac{16\pi G}{3}H\rho - \frac{2}{3}H\Lambda + 2H\left(-H^2 - \frac{4\pi G}{3}\rho + \frac{1}{3}\Lambda\right) + 8\pi GH\rho\right] \\ &= 0\,. \end{aligned} \tag{3.2.10}$$

It follows that

$$H^2 + \frac{K}{a^2} = \frac{1}{3}(8\pi G\rho + \Lambda)\,. \tag{3.2.11}$$

In general relativity, K/a^2 is 1/6 of the intrinsic curvature of the surfaces of homogeneity.

3.2.2 Comoving coordinates

A comoving observer in the background (i.e. unperturbed) cosmology has velocity $d\mathbf{r}/dt = H(t)\mathbf{r}$ hence position $\mathbf{r} = a(t)\mathbf{x}$ where $\mathbf{x}$ is a constant. Rather than labelling events by t and $\mathbf{r}$, it is convenient to use t and $\mathbf{x}$, where $\mathbf{x}$ are *comoving spatial coordinates*: $\mathbf{x} = \mathbf{r}/a(t)$. Note these are *Lagrangian coordinates* in the background but not in the perturbed model.

Derivatives transform as follows:

$$\begin{aligned}\left(\frac{\partial}{\partial t}\right)_{\mathbf{r}} &= \left(\frac{\partial}{\partial t}\right)_{\mathbf{x}} + \left(\frac{\partial \mathbf{x}}{\partial t}\right)_{\mathbf{r}} \cdot \boldsymbol{\nabla}_{\mathbf{x}} \\ &= \left(\frac{\partial}{\partial t}\right)_{\mathbf{x}} - H(t)\mathbf{x}\cdot\boldsymbol{\nabla}\,, \end{aligned} \tag{3.2.12}$$

where we use $\boldsymbol{\nabla}$ to denote the gradient with respect to $\mathbf{x}$ at fixed t; and

$$\boldsymbol{\nabla}_{\mathbf{r}} = a^{-1}\boldsymbol{\nabla}\,. \tag{3.2.13}$$

In what follows, ∂_t should be understood as being taken at fixed $\mathbf{x}$.

3.2.3 Perturbation analysis

We now perturb ρ, $\mathbf{u}$ and Φ about their background values:

$$\rho \to \bar{\rho}(t) + \delta\rho \equiv \bar{\rho}(t)(1+\delta) \tag{3.2.14}$$

$$P \to \bar{P}(t) + \delta P \tag{3.2.15}$$

$$\mathbf{u} \to a(t)H(t)\mathbf{x} + \mathbf{v} \tag{3.2.16}$$

$$\Phi \to \bar{\Phi}(\mathbf{x},t) + \phi\,. \tag{3.2.17}$$

Here, δ is the *fractional overdensity* in the fluid and ϕ the perturbed gravitational potential. Since, for a particle in the fluid,

$$\frac{d\mathbf{r}}{dt} = \frac{d(a\mathbf{x})}{dt} = aH\mathbf{x} + a\frac{d\mathbf{x}}{dt} = \mathbf{u}\,, \tag{3.2.18}$$

we see that $ad\mathbf{x}/dt = \mathbf{v}$, so the *peculiar velocity* $\mathbf{v}$ describes changes in the comoving coordinates of fluid elements in time (i.e. departures from the background Hubble flow).

The continuity equation (3.2.1) becomes (on using Eq. 3.2.12)

$$(1+\delta)\partial_t\bar{\rho} - H\bar{\rho}\mathbf{x}\cdot\boldsymbol{\nabla}\delta + \bar{\rho}\partial_t\delta + \frac{\bar{\rho}}{a}\boldsymbol{\nabla}\cdot[(1+\delta)(aH\mathbf{x}+\mathbf{v})] = 0\,. \tag{3.2.19}$$

Gathering terms that are zeroth, first and second-order in products of perturbed quantities gives

$$\underbrace{\partial_t\bar{\rho}+3\bar{\rho}H}_{\text{0th-order}}+\underbrace{(\partial_t\bar{\rho}+3\bar{\rho}H)\delta+\bar{\rho}\partial_t\delta+\frac{\bar{\rho}}{a}\boldsymbol{\nabla}\cdot\mathbf{v}}_{\text{1st-order}}+\underbrace{\frac{\bar{\rho}}{a}(\mathbf{v}\cdot\boldsymbol{\nabla}\delta+\delta\boldsymbol{\nabla}\cdot\mathbf{v})}_{\text{2nd-order}}=0\,. \tag{3.2.20}$$

The background equation (3.2.7) sets the zero-order part to zero. In *linear perturbation theory*, we assume the perturbations are small enough (and their spatial derivatives) that we can ignore the second-order part, so that

$$\partial_t\delta+\frac{1}{a}\boldsymbol{\nabla}\cdot\mathbf{v}=0\,. \tag{3.2.21}$$

Exercise: show that Eqs (3.2.2) and (3.2.3) linearise to

$$\partial_t\mathbf{v}+H\mathbf{v}=-\frac{1}{a\bar{\rho}}\boldsymbol{\nabla}\delta P-\frac{1}{a}\boldsymbol{\nabla}\phi \tag{3.2.22}$$

$$\boldsymbol{\nabla}^2\phi=4\pi Ga^2\bar{\rho}\delta\,. \tag{3.2.23}$$

Scalar/vector decomposition

We can always decompose the vector $\mathbf{v}$ as

$$\mathbf{v}=\underbrace{\boldsymbol{\nabla}v}_{\text{scalar part}}+\underbrace{\mathbf{v}_\perp}_{\text{vector part}}\,, \tag{3.2.24}$$

where $\boldsymbol{\nabla}\cdot\mathbf{v}_\perp=0$. It follows from Eq. (3.2.21) that the vector part of $\mathbf{v}$ does not lead to clumping of the matter. Since $\boldsymbol{\nabla}\times\mathbf{v}=\boldsymbol{\nabla}\times\mathbf{v}_\perp$, $\mathbf{v}_\perp$ describes the vorticity of the fluid – recalling that $\boldsymbol{\nabla}_\mathbf{r}=a^{-1}\boldsymbol{\nabla}$, the physical vorticity $\boldsymbol{\nabla}_\mathbf{r}\times\mathbf{u}=a^{-1}\boldsymbol{\nabla}\times\mathbf{v}_\perp$. In linear theory, the scalar and vector parts decouple. For example, consider the (comoving) curl of the perturbed Euler equation (3.2.22),

$$\boldsymbol{\nabla}\times\partial_t\mathbf{v}=\partial_t(\boldsymbol{\nabla}\times\mathbf{v}_\perp)=-H\boldsymbol{\nabla}\times\mathbf{v}_\perp\,. \tag{3.2.25}$$

It follows that $\boldsymbol{\nabla}\times\mathbf{v}_\perp$ decays as $1/a$ in an expanding universe so the vorticity falls as $1/a^2$. This decay of the vorticity is consistent with the circulation theorem, $\oint\mathbf{u}\cdot d\mathbf{r}=$ const. for a path comoving with the fluid. For general initial conditions, the peculiar velocity approaches a gradient at late times and the vector modes can be neglected. For initial conditions from inflation, vector modes are not excited in the first place. They are, however, important in models with continual sourcing of perturbations by cosmic defects.

3.2.4 Jeans' length

The time derivative of the perturbed continuity equation (3.2.21) gives

$$\partial_t^2\delta - \frac{1}{a}H\boldsymbol{\nabla}\cdot\mathbf{v} + \frac{1}{a}\boldsymbol{\nabla}\cdot\partial_t\mathbf{v} = 0. \tag{3.2.26}$$

Combining with the perturbed Euler equation (3.2.22) and the Poisson equation (3.2.23), we find

$$\begin{aligned}
\partial_t^2\delta - \frac{1}{a}H\boldsymbol{\nabla}\cdot\mathbf{v} - \frac{1}{a}\boldsymbol{\nabla}\cdot\left(H\mathbf{v} + \frac{1}{a\bar\rho}\boldsymbol{\nabla}\delta P + \frac{1}{a}\boldsymbol{\nabla}\phi\right) = 0 \\
\Rightarrow \quad \partial_t^2\delta - \frac{2}{a}H\boldsymbol{\nabla}\cdot\mathbf{v} - \frac{1}{a^2\bar\rho}\boldsymbol{\nabla}^2\delta P - \frac{1}{a^2}\boldsymbol{\nabla}^2\phi = 0 \\
\Rightarrow \quad \partial_t^2\delta + 2H\partial_t\delta - 4\pi G\bar\rho\delta - \frac{1}{a^2\bar\rho}\boldsymbol{\nabla}^2\delta P = 0\,.
\end{aligned} \tag{3.2.27}$$

This is the fundamental equation for the growth of structure in Newtonian theory. It shows the general competition between infall by gravitational attraction – the $4\pi G\bar\rho\delta$ term – and pressure support, $\boldsymbol{\nabla}^2\delta P$.

Consider a *barotropic* fluid such that $P = P(\rho)$; then

$$\delta P = \frac{\partial P}{\partial \rho}\bar\rho\delta \equiv c_s^2\bar\rho\delta \tag{3.2.28}$$

where c_s^2 is the sound speed. Using this in Eq. (3.2.27), and Fourier expanding so that $\boldsymbol{\nabla}^2 \to -k^2$, gives

$$\partial_t^2\delta + 2H\partial_t\delta + \left(\frac{c_s^2k^2}{a^2} - 4\pi G\bar\rho\right)\delta = 0\,. \tag{3.2.29}$$

This is the equation for a damped (in an expanding universe) oscillator provided that

$$\frac{c_s^2k^2}{a^2} > 4\pi G\bar\rho\,, \tag{3.2.30}$$

and, in this case, the pressure support gives rise to acoustic oscillations (sound waves) in the fluid. However, for $c_s^2k^2/a^2 < 4\pi G\bar\rho$, the system is unstable to gravitational accretion. Perturbations with *proper wavelength* $2\pi a/k$ exceeding the (proper) *Jeans' wavelength*,

$$\lambda_{\rm J} \equiv c_s\sqrt{\frac{\pi}{G\bar\rho}}\,, \tag{3.2.31}$$

are gravitationally unstable, while on smaller scales pressure supports oscillations.

The Jeans' length is roughly the radius R of a region of background density $\bar\rho$ such the free-fall time, $t_{\rm ff}$, equals the sound-crossing time, $t_{\rm sound}$. To see this, note that

$t_{\rm ff} \sim R/v_{\rm ff}$, where $v_{\rm ff}$ is the free-fall speed. An object falling from rest at infinity onto the surface of the mass $M \sim \bar{\rho}R^3$ has speed $v_{\rm ff} = \sqrt{GM/R}$ so

$$t_{\rm ff} \sim \frac{R}{\sqrt{GM/R}} \sim \frac{1}{\sqrt{G\bar{\rho}}}\,. \tag{3.2.32}$$

The sound-crossing time is simply $t_{\rm sound} = R/c_s$ and this equals the free-fall time for $R \sim c_s/\sqrt{G\bar{\rho}} \sim \lambda_{\rm J}$. Fluctuations larger than the Jeans' length do not have time for pressure to resist gravitational infall since the time to infall is less than the time it takes to propagate a pressure disturbance (i.e. a sound wave) across the perturbation. Note, finally, that the free-fall time is roughly the Hubble time, $1/H$, when curvature and dark energy are negligible.

3.2.5 Applications to cold dark matter

Solutions in an Einstein-de Sitter phase

After matter-radiation equality, but well before dark energy comes to dominate, our universe is well described by an Einstein-de Sitter model having $\bar{P} \approx 0$ and zero curvature or Λ. Scales of cosmological interest are much larger than the Jeans' scale for the baryons and so both CDM fluctuations and those for the baryons have the same dynamical equations. We shall show shortly that quickly after recombination, the fractional overdensity in the baryons, δ_b, approaches that in the CDM, δ_c, and the matter behaves like a single pressure-free fluid with total density contrast

$$\delta_m = \frac{\bar{\rho}_b\delta_b + \bar{\rho}_c\delta_c}{\bar{\rho}_b + \bar{\rho}_c} \approx \delta_c\,. \tag{3.2.33}$$

Since $H^2 \propto \bar{\rho} \propto a^{-3}$, we have $a \propto t^{2/3}$ and so $H = 2/(3t)$ and $4\pi G\bar{\rho} = 2/(3t^2)$. Equation (3.2.27) then gives the evolution of the density fluctuations in the pressure-free matter as

$$\partial_t^2\delta_m + \frac{4}{3t}\partial_t\delta_m - \frac{2}{3t^2}\delta_m = 0\,. \tag{3.2.34}$$

Trying solutions like t^p gives independent solutions $\delta_m \propto t^{-1}$ and $\delta_m \propto t^{2/3} \propto a$. The *growing-mode* solution of the density contrast therefore grows like the scale factor. Note that here, in an expanding universe, gravitational attraction has given rise to power-law growth of δ to be compared to the exponential growth predicted in a non-expanding model. The Poisson equation (3.2.23) tells us that the gravitational potential is constant since, in Fourier space,

$$-k^2\phi = 4\pi Ga^2 \underbrace{\bar{\rho}}_{\propto a^{-3}}\underbrace{\delta}_{\propto a} = \text{const.} \tag{3.2.35}$$

The Meszaros effect

The Meszaros effect describes the way that CDM grows only logarithmically on scales inside the sound horizon during radiation domination. Generally, CDM (or anything else) feels the gravity of all clustered components so Eq. (3.2.27) generalises to the ith component of a set of non-interacting (except through gravity) fluids as

$$\partial_t^2 \delta_i + 2H\partial_t \delta_i - 4\pi G \sum_j \bar{\rho}_j \delta_j - \frac{1}{a^2 \bar{\rho}_i} \boldsymbol{\nabla}^2 \delta P_i = 0\,. \tag{3.2.36}$$

Specialising to pressure-free CDM we have

$$\partial_t^2 \delta_c + 2H\partial_t \delta_c - 4\pi G \sum_j \bar{\rho}_j \delta_j = 0\,. \tag{3.2.37}$$

Our Newtonian treatment at least makes it plausible that the Jeans' length for perturbations in the radiation fluid (for which $c_s = 1/\sqrt{3}$) during radiation domination is of the order of the Hubble radius. Radiation fluctuations on scales smaller than this therefore oscillate as sound waves and their time-averaged density contrast vanishes (we shall show this properly when we develop relativistic perturbation theory). It follows that the CDM is essentially the only clustered component during the acoustic oscillations of the radiation, and so

$$\partial_t^2 \delta_c + \frac{1}{t}\partial_t \delta_c - 4\pi G \bar{\rho}_c \delta_c = 0\,, \tag{3.2.38}$$

where we used $a \propto t^{1/2}$ and so $H = 1/(2t)$. Since δ_c evolves only on cosmological timescales (it has no pressure support for it to do otherwise),

$$\partial_t^2 \delta_c \sim H^2 \delta_c \gg 4\pi G \bar{\rho}_c \delta_c \tag{3.2.39}$$

during radiation domination, as $\bar{\rho}_r \gg \bar{\rho}_c$. We can therefore ignore the last term in Eq. (3.2.38) compared to the others and we have solutions with $\delta_c = \text{const.}$ and $\delta_c \propto \ln t$. We see that the rapid expansion due to the effectively unclustered radiation reduces the growth of δ_c to only logarithmic.

Late-time suppression of structure formation by Λ

At late times, the dominant clustered component is the matter and we have

$$\partial_t^2 \delta_m + 2H\partial_t \delta_m - 4\pi G \bar{\rho}_m \delta_m = 0\,. \tag{3.2.40}$$

In matter domination, this reduces to Eq. (3.2.34) and δ_m grows like a, but when Λ comes to dominate $a \propto e^{t\sqrt{\Lambda/3}}$ and $H \approx \text{const.}$ It follows that $4\pi G \bar{\rho}_m \ll H^2$ (currently $4\pi G \bar{\rho}_m / H^2 \sim 0.37$) and

$$\partial_t^2 \delta_m + 2H\partial_t \delta_m \approx 0\,. \tag{3.2.41}$$

The solutions of this are $\delta_m = \text{const.}$ or $\delta_m \propto e^{-2t\sqrt{\Lambda/3}} \propto a^{-2}$ and Λ suppresses the growth of structure. Note also that a constant density contrast implies that the gravitational potential decays as $a^2\bar{\rho}_m \propto a^{-1}$. This leaves an imprint in the CMB called the *integrated Sachs-Wolfe effect* (see later).

Evolution of baryon fluctuations after decoupling

Before decoupling, the baryon dynamics is linked to that of the radiation by efficient (Compton) scattering. On sub-Hubble scales, δ_b oscillates like the radiation but, after matter-radiation equality, δ_c grows like a. It follows that just after decoupling, $\delta_c \gg \delta_b$. Subsequently, the baryons fall into the potential wells sourced mainly by the CDM and $\delta_b \to \delta_c$ as we shall now show.

Ignoring baryon pressure and Λ, the coupled dynamics of the baryon and CDM fluids after decoupling is approximately given by

$$\partial_t^2\delta_b + \frac{4}{3t}\partial_t\delta_b = 4\pi G(\bar{\rho}_b\delta_b + \bar{\rho}_c\delta_c) \tag{3.2.42}$$

$$\partial_t^2\delta_c + \frac{4}{3t}\partial_t\delta_c = 4\pi G(\bar{\rho}_b\delta_b + \bar{\rho}_c\delta_c)\,. \tag{3.2.43}$$

We can decouple these equations by using normal coordinates δ_m (see equation 3.2.33) and $\Delta \equiv \delta_c - \delta_b$. Then

$$\partial_t^2\Delta + \frac{4}{3t}\partial_t\Delta = 0 \quad \Rightarrow \quad \Delta = \text{const. or } \Delta \propto t^{-1/3}\,, \tag{3.2.44}$$

while δ_m follows Eq. (3.2.34) and has solutions $\propto t^{-1}$ and $t^{2/3}$. Since

$$\frac{\delta_c}{\delta_b} = \frac{\bar{\rho}_m\delta_m + \bar{\rho}_b\Delta}{\bar{\rho}_m\delta_m - \bar{\rho}_c\Delta} \to \frac{\delta_m}{\delta_m} = 1\,, \tag{3.2.45}$$

we see that δ_b approaches δ_c.

The non-zero initial value of δ_b at decoupling, and, more importantly $\partial_t\delta_b$, leaves a small imprint in the late-time δ_m that oscillates with scale. These *baryon acoustic oscillations* have recently been detected in the clustering of galaxies (see Fig. 3).

3.3 Introduction to relativistic perturbation theory

The Newtonian treatment of Section 3.2 is inadequate on scales larger than the Hubble radius, and for relativistic fluids (like tightly-coupled radiation). The correct description requires a full general-relativistic treatment.

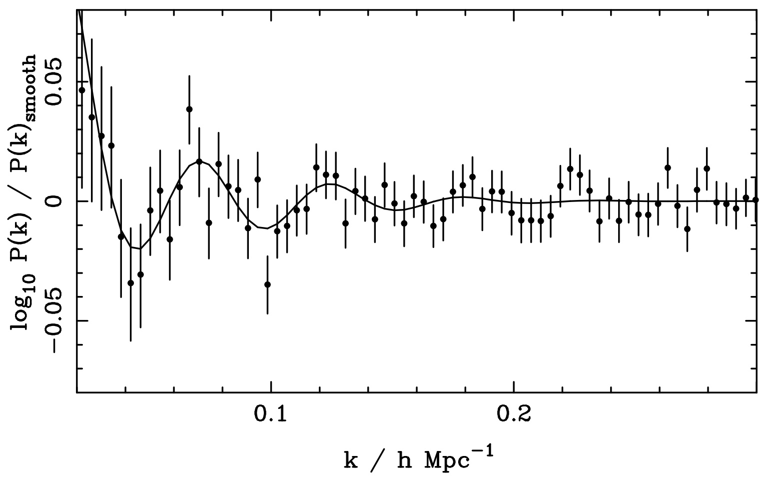

Figure 3: Ratio of the matter power spectrum to a smooth spectrum (i.e. a model with no baryons) showing the expected baryon acoustic oscillations. Credit: Percival et al.

The basic idea of relativistic perturbation theory is straightforward: perturb the metric and stress-energy tensor in the Einstein equations about their Friedmann-Robertson-Walker (FRW) forms, and, for linear perturbations, drop products of small quantities. We then solve the coupled system of equations

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu} + \Lambda \delta g_{\mu\nu} \, . \tag{3.3.1}$$

Even in linear theory there are some technical complexities due to the *gauge freedom* inherent in general relativity (i.e. the freedom over choice of coordinates). For this reason we shall only give an introduction to relativistic perturbation theory here; for a full treatment see the recommended textbooks or attend the Lent-term course *Advanced Cosmology*.

We begin by recalling the background metric and dynamics. The spatially-flat FRW metric is

$$ds^2 = a^2(\eta)(d\eta^2 - \delta_{ij} dx^i dx^j) = a^2 \eta_{\mu\nu} dx^\mu dx^\nu \, . \tag{3.3.2}$$

To avoid unnecessary complications, we shall only consider flat ($K = 0$) universes here. The Friedmann equations for such models are, in conformal time,

$$\mathcal{H}^2 = \frac{1}{3} a^2 (8\pi G \rho + \Lambda) \tag{3.3.3}$$

$$\dot{\mathcal{H}} = \frac{1}{6} a^2 [2\Lambda - 8\pi G(\rho + 3P)] \, , \tag{3.3.4}$$

where $\mathcal{H}$ is the conformal Hubble parameter, $\mathcal{H} \equiv \partial_\eta a / a = aH$, and overdots denote differentiation with respect to conformal time.

3.3.1 Metric perturbations

The most general perturbation to the background metric is

$$ds^2 = a^2(\eta)\left\{(1+2\psi)d\eta^2 - 2B_i dx^i d\eta - [(1-2\phi)\delta_{ij} + 2E_{ij}]\, dx^i dx^j\right\} . \qquad (3.3.5)$$

The following points should be noted.

- The quantities ψ and ϕ are scalar functions of η and x^i though, as we shall see shortly, they are *not* Lorentz scalars.
- We have chosen Cartesian spatial coordinates in the background, but, had we chosen differently (e.g. spherical-polar coordinates), the perturbation B_i would have transformed like a three-vector. To be specific, under η-independent relabelling of spatial coordinates,

$$x^i \to x'^i, \qquad B_i \to \frac{\partial x^j}{\partial x'^i} B_j . \qquad (3.3.6)$$

- The perturbation E_{ij} is a symmetric ($E_{ij} = E_{ji}$) and trace-free ($\delta^{ij}E_{ij} = 0$) three-tensor, i.e. under

$$x^i \to x'^i, \qquad E_{ij} \to \frac{\partial x^k}{\partial x'^i}\frac{\partial x^l}{\partial x'^j} E_{kl} . \qquad (3.3.7)$$

  The background spatial metric $-a^2\delta_{ij}$ transforms similarly.

We shall adopt the useful convention that *Latin indices on spatial vectors and tensors are raised and lowered with* δ_{ij}. For example, $B^i \equiv \delta^{ij}B_j$ and $E^i{}_j = \delta^{ik}E_{kj}$.

Scalar, vector and tensor decomposition

In Section 3.2.3 we decomposed the peculiar velocity into scalar and vector parts. It is convenient to use the same decomposition here, so, for example,

$$B_i = \underbrace{\partial_i B}_{\text{scalar part}} + \underbrace{B_i^T}_{\text{vector part}} , \qquad (3.3.8)$$

where the vector part is transverse (divergence-free), $\delta^{ij}\partial_j B_i^T$. A similar decomposition works for symmetric, trace-free 3-tensors:

$$E_{ij} = \underbrace{\partial_{\langle i}\partial_{j\rangle}E}_{\text{scalar part}} + \underbrace{\partial_{(i}E_{j)}^T}_{\text{vector part}} + \underbrace{E_{ij}^T}_{\text{tensor part}} , \qquad (3.3.9)$$

where

$$\partial_{\langle i}\partial_{j\rangle}E \equiv \partial_i\partial_j E - \frac{1}{3}\delta_{ij}\nabla^2 E \tag{3.3.10}$$

is trace-free; E_i^T is transverse; and E_{ij}^T is symmetric, trace-free and transverse, $\delta^{ik}\partial_k E_{ij}^T = 0$. This decomposition is unique in Euclidean space for smooth, bounded E_{ij} that decay at infinity. A formal proof can be found in J.M. Stewart, *Class. Quantum Grav.* 7 (1990) 1169. Here we shall just give a simple *non-examinable* demonstration by constructing the scalar, vector and tensor parts in Fourier space.

Non-examinable demonstration of scalar, vector and tensor decomposition: Assuming the Fourier transform of E_{ij} exists, we have

$$E_{ij}(\mathbf{k}) = \int \frac{d^3\mathbf{x}}{(2\pi)^{3/2}} E_{ij}(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \,. \tag{3.3.11}$$

Consider decomposing $E_{ij}(\mathbf{k})$ onto a basis of trace-free tensors. We start with a basis of vectors $(\hat{\mathbf{k}}, \mathbf{m}, \mathbf{m}^*)$ where $\mathbf{m}$ is a complex vector perpendicular to $\hat{\mathbf{k}}$ satisfying $\mathbf{m}\cdot\mathbf{m} = 0$ and $\mathbf{m}\cdot\mathbf{m}^* = 1$, and $\mathbf{m}^*$ is the complex conjugate of $\mathbf{m}$. For example, if $\hat{\mathbf{k}}$ were along the z-axis, we could take $\mathbf{m} = (\hat{\mathbf{x}} + i\hat{\mathbf{y}})/\sqrt{2}$. Taking all possible trace-free, symmetric tensor products of pairs of this basis, we obtain a basis on which $E_{ij}(\mathbf{k})$ can be expanded:

$$\begin{aligned}
M_{ij}^{(0)}(\hat{\mathbf{k}}) &= \hat{k}_{\langle i}\hat{k}_{j\rangle} \\
M_{ij}^{(1)}(\hat{\mathbf{k}}) &= \hat{k}_{(i}m_{j)} \\
M_{ij}^{(-1)}(\hat{\mathbf{k}}) &= \hat{k}_{(i}m^*_{j)} \\
M_{ij}^{(2)}(\hat{\mathbf{k}}) &= m_i m_j \\
M_{ij}^{(-2)}(\hat{\mathbf{k}}) &= m_i^* m_j^* \,.
\end{aligned} \tag{3.3.12}$$

Note that the sixth possible pairing

$$m_{\langle i}m^*_{j\rangle} = -\frac{1}{2}M_{ij}^{(0)} - \frac{1}{2}M_{ij}^{(2)} + \frac{1}{2}M_{ij}^{(-2)} \,, \tag{3.3.13}$$

and so is not linearly independent. The five-dimensional basis is as expected since their are five functional degrees of freedom in the trace-free, symmetric E_{ij}.

We can now expand $E_{ij}(\mathbf{k})$ as

$$E_{ij}(\mathbf{k}) = \sum_n E^{(n)}(\mathbf{k}) M_{ij}^{(n)}(\hat{\mathbf{k}}) \,. \tag{3.3.14}$$

Consider the $n = 0$ term:

$$\begin{aligned}
\int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \hat{k}_{\langle i}\hat{k}_{j\rangle} E^{(0)}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} &= -\left(\partial_i\partial_j - \frac{1}{3}\nabla^2\right) \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{E^{(0)}(\mathbf{k})}{k^2} e^{i\mathbf{k}\cdot\mathbf{x}} \\
&= \partial_{\langle i}\partial_{j\rangle} E \,,
\end{aligned} \tag{3.3.15}$$

which gives the scalar part of E_{ij}. For the $n=1$ term,

$$\int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \hat{k}_{(i} m_{j)} E^{(1)}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} = -i\partial_{(i} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} m_{j)}(\hat{\mathbf{k}}) E^{(1)}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}\,, \tag{3.3.16}$$

so, combining the $n=\pm 1$ terms we find the vector-mode part of E_{ij}:

$$-i\partial_{(i} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left[m_{j)}(\hat{\mathbf{k}}) E^{(1)}(\mathbf{k}) + m^*_{j)}(\hat{\mathbf{k}}) E^{(-1)}(\mathbf{k}) \right] e^{i\mathbf{k}\cdot\mathbf{x}} = \partial_{(i} E^T_{j)}\,. \tag{3.3.17}$$

The vector E^T_i is transverse since $\hat{\mathbf{k}}\cdot\mathbf{m}(\hat{\mathbf{k}})=0$. Finally, for the $n=2$ terms we have

$$\int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left[m_i(\hat{\mathbf{k}}) m_j(\hat{\mathbf{k}}) E^{(2)}(\mathbf{k}) + m^*_i(\hat{\mathbf{k}}) m^*_j(\hat{\mathbf{k}}) E^{(-2)}(\mathbf{k}) \right] = E^T_{ij}\,, \tag{3.3.18}$$

which is symmetric and trace-free by construction, and is transverse since, again, $\hat{\mathbf{k}}\cdot\mathbf{m}(\hat{\mathbf{k}})=0$.

If we count the functional degrees of freedom in the scalar, vector and tensor parts, the original five degrees of freedom in E_{ij} are re-distributed as follows.

- Scalar – one degree of freedom $[E(\mathbf{x},\eta)]$.
- Vector – two degrees of freedom since E^T_i has three components, but once constraint (the divergence vanishes).
- Tensor – two degrees of freedom since E^T_{ij} has five components but $\delta^{ik}\partial_k E^T_{ij}$ is three constraints (one for each component).

Given the uniqueness of the decomposition, in any 3-tensor equation the scalar, vector and tensor parts of the left and right-hand sides must be equal and, in linear theory, this means that the perturbation types decouple completely. As in Newtonian theory, scalar perturbations describe clumping of matter while vector modes describe vortical motions. The new feature in the relativistic treatment here is the tensor modes describing *gravitational waves*. We shall mostly be concerned with scalar perturbations here.

Orthonormal frame vectors

Finally, it will prove useful to construct explicitly an *orthonormal frame* of 4-vectors, $(E_0)^\mu$ and $(E_i)^\mu$, in the perturbed metric. Take the timelike $(E_0)^\mu$ to be the 4-velocity

u^μ of an observer at rest relative to the coordinate system. It follows that $(E_0)^\mu$ must be parallel to δ_0^μ and normalising gives, at linear order,

$$(E_0)^\mu = a^{-1}(1-\psi)\delta_0^\mu \,, \tag{3.3.19}$$

since then

$$g_{\mu\nu}(E_0)^\mu (E_0)^\nu = a^{-2}(1-2\psi)g_{00} = a^{-2}(1-2\psi)a^2(1+2\psi) = 1\,. \tag{3.3.20}$$

Note how we have dropped second-order terms (i.e. products of perturbations) here.

The spacelike $(E_i)^\mu$ are a little more involved since, generally, the coordinate vectors δ_i^μ are not orthogonal to u^μ unless $B_i = 0$. The following construction has the required properties:

$$(E_i)^\mu = a^{-1}\left[B_i\delta_0^\mu + (1+\phi)\delta_i^\mu - E_i{}^j\delta_j^\mu\right] \,. \tag{3.3.21}$$

We can easily check that these are orthogonal to $(E_0)^\mu$:

$$\begin{aligned} g_{\mu\nu}(E_0)^\mu (E_i)^\nu &= a^{-1}(1-\psi)g_{0\nu}(E_i)^\nu \\ &= a^{-2}(1-\psi)\left[g_{00}B_i + g_{0i}(1+\phi) - g_{0j}E_i{}^j\right] \\ &= B_i - B_i = 0\,. \end{aligned} \tag{3.3.22}$$

Exercise: show that $g_{\mu\nu}(E_i)^\mu (E_j)^\nu = -\delta_{ij}$.

3.3.2 Matter perturbations

Recall that the energy and momentum of the matter is described by the stress-energy tensor. In an orthonormal frame, the components of $T^{\mu\nu}$ are

$$\begin{aligned} &T^{\hat{0}\hat{0}} = \bar{\rho}(\eta) + \delta\rho && \text{energy density} \\ &T^{\hat{0}\hat{i}} = q^i && \text{momentum density} \\ &T^{\hat{i}\hat{j}} = [\bar{P}(\eta) + \delta P]\delta^{ij} - \Pi^{ij} && \text{flux of } i\text{th component of 3-momentum} \\ & && \text{along } j\text{th direction}\,, \end{aligned} \tag{3.3.23}$$

where Π^{ij} is the trace-free *anisotropic stress*. We shall use the perturbations $\delta\rho \equiv \bar{\rho}(\eta)(1+\delta)$, δP, q^i and Π^{ij} to describe the perturbations to the matter and fields present, and their components are defined on the orthonormal frame in Eqs (3.3.19) and (3.3.21). However, it will be convenient to construct the coordinate components of $T^{\mu\nu}$ which are given by

$$T^{\mu\nu} = (E_\alpha)^\mu (E_\beta)^\nu T^{\hat{\alpha}\hat{\beta}} \,. \tag{3.3.24}$$

We start with T^{00}:

$$\begin{aligned} T^{00} &= (E_0)^0(E_0)^0 T^{\hat{0}\hat{0}} + 2(E_0)^0(E_i)^0 T^{\hat{0}\hat{i}} + (E_i)^0(E_j)^0 T^{\hat{i}\hat{j}} \\ &= a^{-2}(1-2\psi)\bar{\rho}(1+\delta) + O(2) + O(2) \\ &= a^{-2}\bar{\rho}(1+\delta-2\psi)\,. \end{aligned} \tag{3.3.25}$$

For T^{0i}, we have

$$\begin{aligned} T^{0i} &= (E_0)^0(E_0)^i T^{\hat{0}\hat{0}} + (E_0)^0(E_j)^i T^{\hat{0}\hat{j}} + (E_j)^0(E_0)^i T^{\hat{0}\hat{j}} + (E_j)^0(E_k)^i T^{\hat{j}\hat{k}} \\ &= 0 + a^{-2}\delta^i_j q^j + 0 + a^{-2}B_j\delta^i_k \bar{P}\delta^{jk} \\ &= a^{-2}(q^i + \bar{P}B^i)\,. \end{aligned} \tag{3.3.26}$$

Finally, for T^{ij},

$$\begin{aligned} T^{ij} &= (E_0)^i(E_0)^j T^{\hat{0}\hat{0}} + (E_0)^i(E_k)^j T^{\hat{0}\hat{k}} + (E_k)^i(E_0)^j T^{\hat{k}\hat{0}} + (E_k)^i(E_l)^j T^{\hat{k}\hat{l}} \\ &= 0+0+0+a^{-2}\left[(1+\phi)\delta^i_k - {E_k}^i\right]\left[(1+\phi)\delta^j_l - {E_l}^j\right]\left[(\bar{P}+\delta P)\delta^{kl} - \Pi^{kl}\right] \\ &= a^{-2}\left[\bar{P}\delta^{ij} + (2\bar{P}\phi + \delta P)\delta^{ij} - 2\bar{P}E^{ij} - \Pi^{ij}\right]\,. \end{aligned} \tag{3.3.27}$$

Note how these components mix the perturbations to the stress-energy tensor with the metric perturbations. Things look neater in term of the mixed coordinate components:

$$\begin{aligned} {T^0}_0 = g_{\mu 0}T^{0\mu} &= g_{00}T^{00} + g_{0i}T^{0i} \\ &= a^2(1+2\psi)a^{-2}\bar{\rho}(1-2\psi+\delta) + O(2) \\ &= \bar{\rho}(1+\delta)\,. \end{aligned} \tag{3.3.28}$$

Exercise: Show that the other mixed components are

$$ {T^i}_0 = q^i \tag{3.3.29}$$

$$ {T^i}_j = -(\bar{P}+\delta P)\delta^i_j + {\Pi^i}_j\,, \tag{3.3.30}$$

where ${\Pi^i}_j \equiv \delta_{jk}\Pi^{ik}$.

We can gain some insight into the momentum density by considering a perfect fluid with 4-velocity u^μ which is a small perturbation to the 4-velocity of observers at rest in our coordinates. The stress-energy tensor for a perfect fluid is

$$ {T^\mu}_\nu = (\rho + P)u^\mu u_\nu - P\delta^\mu_\nu\,. \tag{3.3.31}$$

Since the 4-velocity $u^\mu = dx^u/d\tau$, where τ is the proper time, and $g_{\mu\nu}u^\mu u^\nu = 1$, we have

$$\begin{aligned}
1 &= g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \\
&= g_{\mu\nu}\left(\frac{d\eta}{d\tau}\right)^2\frac{dx^\mu}{d\eta}\frac{dx^\nu}{d\eta} \\
&= \left(\frac{d\eta}{d\tau}\right)^2\left(g_{00} + 2g_{0i}\frac{dx^i}{d\eta} + g_{ij}\frac{dx^i}{d\eta}\frac{dx^j}{d\eta}\right) .
\end{aligned} \tag{3.3.32}$$

If we write the *coordinate velocity* $dx^i/d\eta = v^i$ and assume this is a small perturbation, then Eq. (3.3.32) reduces to

$$1 = \left(\frac{d\eta}{d\tau}\right)^2 g_{00} = a^2(1+2\psi)\left(\frac{d\eta}{d\tau}\right)^2 \qquad \Rightarrow \qquad \frac{d\eta}{d\tau} = \frac{1}{a}(1-\psi) \tag{3.3.33}$$

at linear order. The components of the fluid's 4-velocity are then

$$u^\mu = a^{-1}[1-\psi, v^i] , \tag{3.3.34}$$

and

$$u_0 = g_{00}u^0 + g_{0i}u^i = a^2(1+2\psi)a^{-1}(1-\psi) + O(2) = a(1+\psi) \tag{3.3.35}$$

$$u_i = g_{i0}u^0 + g_{ij}u^j = -a^2B_ia^{-1} - a^2\delta_{ij}a^{-1}v^j = -a(B_i + v_i) . \tag{3.3.36}$$

Using these expressions for the components u^μ and u_μ in Eq. (3.3.31), we find

$$q^i = T^i{}_0 = (\rho+P)a^{-1}v^ia(1+\psi) = (\rho+P)v^i . \tag{3.3.37}$$

Generally, we can define an effective peculiar velocity of the matter (for any component, or the total) by

$$q^i \equiv (\bar\rho + \bar P)v^i . \tag{3.3.38}$$

We end this section by noting that the scalar, vector and tensor decomposition can also be applied to the perturbations to the stress-energy tensor: $\delta\rho$ and δP have scalar parts only, q^i has scalar and vector parts, and Π^{ij} has scalar, vector and tensor parts.

3.3.3 Gauge transformations

The value of the perturbation to a quantity, even one that is a Lorentz scalar, will change under a coordinate transformation. For a given event in the real, clumpy universe, changing coordinates alters the point in the background model that is associated with that event and, if a quantity evolves in the background, the perturbation to that quantity will change. We need to be aware of this *gauge freedom* and develop strategies for handling it.

Consider *small* changes in coordinates

$$\tilde{\eta} = \eta + T(\eta, x^i)\,, \quad \tilde{x}^i = x^i + L^i(\eta, x^j)\,. \tag{3.3.39}$$

For a (Lorentz) scalar field Φ (such as the inflaton field driving inflation), at some event originally labelled by η and x^i, but by $\tilde{\eta}$ and $\tilde{x}^i$ in the new coordinates, the new perturbation *at the same event* is

$$\begin{aligned} \delta\tilde{\Phi} &= \Phi - \bar{\Phi}(\tilde{\eta}) \qquad \text{(since } \bar{\Phi} \text{ is homogeneous)} \\ &= \Phi - \bar{\Phi}(\eta + T) \\ &= \underbrace{\Phi - \bar{\Phi}(\eta)}_{\delta\Phi} - T\dot{\bar{\Phi}}\,, \end{aligned} \tag{3.3.40}$$

to first order in the perturbation to the coordinates. This gives

$$\delta\tilde{\Phi} = \delta\Phi - T\dot{\bar{\Phi}}\,. \tag{3.3.41}$$

Things are a little more complicated for the metric perturbations since $g_{\mu\nu}$ is not a Lorentz scalar. If $\bar{g}_{\mu\nu}$ is the metric in the background, then

$$\begin{aligned} \delta\tilde{g}_{\mu\nu} &= \tilde{g}_{\mu\nu} - \bar{g}_{\mu\nu}(\tilde{\eta}, \tilde{x}^i) \\ &= \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta} - \bar{g}_{\mu\nu}(\tilde{\eta}, \tilde{x}^i) \\ &= \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} \left[\delta g_{\alpha\beta} + \bar{g}_{\alpha\beta}(\eta, x^i)\right] - \bar{g}_{\mu\nu}(\tilde{\eta}, \tilde{x}^i) \\ &= \delta g_{\mu\nu} + \left(\frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} - \delta^\alpha_\mu \delta^\beta_\nu\right) \bar{g}_{\alpha\beta}(\eta, x^i) - T\dot{\bar{g}}_{\mu\nu}(\eta, x^i) - L^i\partial_i \bar{g}_{\mu\nu}(\eta, x^i) \end{aligned} \tag{3.3.42}$$

to linear order. To evaluate this, we require $\partial x^\alpha / \partial \tilde{x}^\mu$ which, considered as a matrix, is the inverse of $\partial \tilde{x}^\mu / \partial x^\alpha$ since

$$\frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial \tilde{x}^\mu}{\partial x^\beta} = \delta^\alpha_\beta\,. \tag{3.3.43}$$

The matrix of derivatives is, from Eq. (3.3.39),

$$\frac{\partial \tilde{x}^\alpha}{\partial x^\mu} = \begin{pmatrix} \partial\tilde{\eta}/\partial\eta & \partial\tilde{\eta}/\partial x^i \\ \partial\tilde{x}^i/\partial\eta & \partial\tilde{x}^i/\partial x^j \end{pmatrix} = \begin{pmatrix} 1 + \dot{T} & \partial_i T \\ \dot{L}^i & \delta^i_j + \partial_j L^i \end{pmatrix}, \tag{3.3.44}$$

where α labels the rows and μ the columns. The inverse of a matrix of the form $\mathsf{I} + \boldsymbol{\Delta}$, were I is the identity and $\boldsymbol{\Delta}$ is a small perturbation, is $\mathsf{I} - \boldsymbol{\Delta}$ to first-order in $\boldsymbol{\Delta}$. It follows that

$$\frac{\partial x^\alpha}{\partial \tilde{x}^\mu} = \begin{pmatrix} \partial\eta/\partial\tilde{\eta} & \partial\eta/\partial\tilde{x}^i \\ \partial x^i/\partial\tilde{\eta} & \partial x^i/\partial\tilde{x}^j \end{pmatrix} = \begin{pmatrix} 1 - \dot{T} & -\partial_i T \\ -\dot{L}^i & \delta^i_j - \partial_j L^i \end{pmatrix}. \tag{3.3.45}$$

We can now substitute into Eq. (3.3.42), and make use of Eq. (3.3.5), to find how the metric perturbations transform. We shall do one case in detail here; the others are left as an exercise. For the 00 component,

$$
\begin{aligned}
\delta\tilde{g}_{00} &= \delta g_{00} + \left(\frac{\partial x^\alpha}{\partial\tilde{\eta}}\frac{\partial x^\beta}{\partial\tilde{\eta}} - \delta^\alpha_0\delta^\beta_0\right)\bar{g}_{\alpha\beta} - T\dot{\bar{g}}_{00} - L^i\partial_i\bar{g}_{00} \\
\Rightarrow \qquad 2a^2\tilde{\psi} &= 2a^2\psi + \left(\frac{\partial\eta}{\partial\tilde{\eta}}\frac{\partial\eta}{\partial\tilde{\eta}} - 1\right)\bar{g}_{00} + 2\frac{\partial\eta}{\partial\tilde{\eta}}\frac{\partial x^i}{\partial\tilde{\eta}}\bar{g}_{0i} + \frac{\partial x^i}{\partial\tilde{\eta}}\frac{\partial x^j}{\partial\tilde{\eta}}\bar{g}_{ij} - T\partial_\eta a^2 \\
&= 2a^2\psi - 2\dot{T}a^2 + O(2) + O(2) - 2T\mathcal{H}a^2\,, \qquad (3.3.46)
\end{aligned}
$$

so we have

$$\tilde{\psi} = \psi - \dot{T} - \mathcal{H}T\,. \qquad (3.3.47)$$

Exercise: By considering the other metric components, show that

$$
\begin{aligned}
\tilde{\phi} &= \phi + \mathcal{H}T + \frac{1}{3}\partial_i L^i \qquad (3.3.48)\\
\tilde{B}_i &= B_i + \partial_i T - \dot{L}_i \qquad (3.3.49)\\
\tilde{E}_{ij} &= E_{ij} - \partial_{\langle i}L_{j\rangle}\,. \qquad (3.3.50)
\end{aligned}
$$

We can also repeat this analysis for the stress-energy tensor. Here, it is more convenient to work with the mixed components which transform as

$$
\begin{aligned}
\delta\tilde{T}^\mu{}_\nu &= \frac{\partial\tilde{x}^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial\tilde{x}^\nu}T^\alpha{}_\beta - \bar{T}^\mu{}_\nu(\eta+T, x^i+L^i) \\
&= \delta T^\mu{}_\nu + \left(\frac{\partial\tilde{x}^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial\tilde{x}^\nu} - \delta^\mu_\alpha\delta^\beta_\nu\right)\bar{T}^\alpha{}_\beta - T\dot{\bar{T}}^\mu{}_\nu - L^i\partial_i\bar{T}^\mu{}_\nu\,, \qquad (3.3.51)
\end{aligned}
$$

where $\bar{T}^\mu{}_\nu$ is the stress-energy tensor in the background. For $T^0{}_0 = \bar{\rho} + \delta\rho$, we have

$$
\begin{aligned}
\delta\tilde{\rho} &= \delta\rho + \left(\frac{\partial\tilde{\eta}}{\partial x^\alpha}\frac{\partial x^\beta}{\partial\tilde{\eta}} - \delta^0_\alpha\delta^\beta_0\right)\bar{T}^\alpha{}_\beta - T\dot{\bar{\rho}} - L^i\partial_i\bar{\rho} \\
&= \delta\rho + \left(\frac{\partial\tilde{\eta}}{\partial\eta}\frac{\partial\eta}{\partial\tilde{\eta}} - 1\right)\bar{\rho} - \frac{\partial\tilde{\eta}}{\partial x^i}\frac{\partial x^j}{\partial\tilde{\eta}}\bar{P}\delta^i_j - T\dot{\bar{\rho}} \\
&= \delta\rho + \left[(1+\dot{T})(1-\dot{T}) - 1\right]\bar{\rho} - O(2) - T\dot{\bar{\rho}}\,, \qquad (3.3.52)
\end{aligned}
$$

so that

$$\delta\tilde{\rho} = \delta\rho - T\dot{\bar{\rho}}\,. \qquad (3.3.53)$$

The remaining transformations are left as an exercise.

Exercise: Show that the pressure perturbation, momentum density and anisotropic stress transform as

$$
\begin{aligned}
\delta\tilde{P} &= \delta P - T\dot{\bar{P}} &(3.3.54)\\
\tilde{q}^i &= q^i + (\bar{\rho} + \bar{P})L^i &(3.3.55)\\
\tilde{\Pi}^i{}_j &= \Pi^i{}_j\,. &(3.3.56)
\end{aligned}
$$

Generally, the perturbations to the stress-energy tensor, $\delta\rho$, δP, q^i and Π^{ij}, change under gauge transformations for two reasons: (i) the transformation changes the 4-velocity of the observer; and (ii) the identified point in the background changes. The density and pressure only change at second-order due to changes in the observer's velocity, but their perturbation changes linearly under gauge-transformations since the density and pressure do not vanish (and evolve) in the background. The momentum density vanishes in the background so its transformation comes solely from the change in the observer's velocity. Indeed, an observer at rest in the new coordinates originally had coordinate 3-velocity $-\dot{L}^i$ and we see from Eq. (3.3.55) that the peculiar velocity of the matter changes by minus this amount (to linear order). The anisotropic stress is invariant under (linear) gauge transformations.

Gauge-invariant variables

Consider scalar perturbations in some gauge, and apply a gauge transformation that has only scalar modes, i.e.

$$\tilde{\eta} = \eta + T\,, \qquad \tilde{x}^i = x^i + \delta^{ij}\partial_j L\,. \tag{3.3.57}$$

The original perturbations to the metric are ψ, ϕ, $\partial_i B$ and $\partial_{\langle i}\partial_{j\rangle}E$. We see from Eqs (3.3.47)–(3.3.50) that the new metric perturbations are also scalar in character with potentials

$$
\begin{aligned}
\tilde{\psi} &= \psi - \dot{T} - \mathcal{H}T &(3.3.58)\\
\tilde{\phi} &= \phi + \mathcal{H}T + \frac{1}{3}\nabla^2 L &(3.3.59)\\
\tilde{B} &= B + T - \dot{L} &(3.3.60)\\
\tilde{E} &= E - L\,. &(3.3.61)
\end{aligned}
$$

There are four functional degrees of freedom in the scalar perturbations to the metric and two gauge functions (T and L) so we expect to be able to construct two *gauge-invariant* quantities (i.e. ones that do not change under gauge transformations). One possibility is the *Bardeen variables*[3]

$$\Psi \equiv \psi + \mathcal{H}(B - \dot{E}) + \dot{B} - \ddot{E} \tag{3.3.62}$$

$$\Phi \equiv \phi - \mathcal{H}(B - \dot{E}) + \frac{1}{3}\nabla^2 E \,. \tag{3.3.63}$$

It is also possible to combine metric and matter variables to form gauge-invariant quantities. One useful combination that we shall use repeatedly is

$$\bar{\rho}\Delta \equiv \delta\rho - 3\mathcal{H}(\bar{\rho} + \bar{P})(B + v)\,, \tag{3.3.64}$$

where $q_i = (\bar{\rho} + \bar{P})\partial_i v$.

Unless we *fix the gauge* completely, *gauge modes* will appear when we solve the perturbed field equations. Gauge modes are apparent perturbations that arise from gauge transformations of the unperturbed background model. For example, a foolish choice of constant-time hypersurfaces in an FRW universe will produce an apparent density perturbation of the form $\delta\rho = -T\dot{\bar{\rho}}$. Such a perturbation is not physical and can be made to vanish by a gauge transformation. By construction, the gauge-invariant variables receive no contributions from gauge modes.

Generally, we can use the gauge freedom inherent in relativistic perturbation theory to simplify the form of our equations, but different choices are convenient for different problems.

3.3.4 Field equations for scalar perturbations in the conformal Newtonian gauge

We shall consider only scalar perturbations here. We can use the two gauge functions T and L to set the metric perturbations E and B to zero. This defines the *conformal Newtonian gauge*

$$ds^2 = a^2(\eta)\left[(1 + 2\psi)d\eta^2 - (1 - 2\phi)\delta_{ij}dx^i dx^j\right]\,. \tag{3.3.65}$$

For perturbations that decay at spatial infinity, the conformal Newtonian gauge is unique (i.e. the gauge is fixed)[4]. In this gauge, the physics appears rather simple since

[3]These were first written down in the classic paper of J.M. Bardeen, *Phys. Rev. D* 22 (1980) 1882.

[4]More generally, a gauge transformation that corresponds to a small, time-dependent but spatially constant boost – i.e. $L^i(\eta)$ and a compensating time translation with $\partial_i T = L_i(\eta)$ to keep the constant-time hypersurfaces orthogonal – will preserve $E_{ij} = 0$ and $B_i = 0$ and hence the form of the metric in Eq. (3.3.65). However, such a transformation would not preserve the decay of the perturbations at infinity.

the hypersurfaces of constant time are orthogonal to the worldlines of observers at rest in the coordinates (since $g_{0i} = 0$) and the induced geometry of the constant-time hypersurfaces is isotropic[5]. Note the similarity of the metric to the usual weak-field limit of general relativity about Minkowski space; we shall see that ϕ plays the role of the gravitational potential.

Perturbed connection coefficients

To derive the field equations, we first require the perturbed connection coefficients. Generally,

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\kappa} \left(\partial_\nu g_{\kappa\rho} + \partial_\rho g_{\kappa\nu} - \partial_\kappa g_{\nu\rho} \right) . \tag{3.3.66}$$

The metric is diagonal so simple to invert:

$$g^{\mu\nu} = \frac{1}{a^2} \begin{pmatrix} 1 - 2\psi & 0 \\ 0 & -(1+2\phi)\delta^{ij} \end{pmatrix} . \tag{3.3.67}$$

Here we shall work out Γ^0_{00} to first order in the perturbations; the remaining terms are left as an exercise:

$$\begin{aligned} \Gamma^0_{00} &= \frac{1}{2} g^{00} (2\partial_\eta g_{00} - \partial_\eta g_{00}) \\ &= \frac{1}{2} g^{00} \partial_\eta g_{00} \\ &= \frac{1}{2a^2} (1 - 2\psi) \partial_\eta [a^2 (1 + 2\psi)] \\ &= \mathcal{H} + \dot{\psi} \, . \end{aligned} \tag{3.3.68}$$

Exercise: Show that the other components of the connection are

$$\Gamma^0_{0i} = \partial_i \psi \tag{3.3.69}$$

$$\Gamma^i_{00} = \delta^{ij} \partial_j \psi \tag{3.3.70}$$

$$\Gamma^0_{ij} = \mathcal{H} \delta_{ij} - \left[\dot{\phi} + 2\mathcal{H}(\phi + \psi) \right] \delta_{ij} \tag{3.3.71}$$

$$\Gamma^i_{j0} = \mathcal{H} \delta^i_j - \dot{\phi} \delta^i_j \tag{3.3.72}$$

$$\Gamma^i_{jk} = -2\delta^i_{(j} \partial_{k)} \phi + \delta_{jk} \delta^{il} \partial_l \phi \, . \tag{3.3.73}$$

---

[5] It can be shown that the relative motion of the coordinate observers is shear and vorticity free.

Stress-energy conservation

The field equations $G_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}$ imply conservation of energy and momentum via the contracted Bianchi identity:

$$\nabla^\mu G_{\mu\nu} = 0 \quad \Rightarrow \quad \nabla^\mu T_{\mu\nu} = 0\,. \tag{3.3.74}$$

This conservation law is the relativistic version of the continuity and Euler equations (3.2.1) and (3.2.2) in Newtonian dynamics. It is more convenient to work with the mixed components $\nabla_\mu T^\mu{}_\nu = 0$ or, explicitly,

$$\partial_\mu T^\mu{}_\nu + \Gamma^\mu_{\mu\rho} T^\rho{}_\nu - \Gamma^\rho_{\mu\nu} T^\mu{}_\rho = 0\,. \tag{3.3.75}$$

Consider first the 0 component which will give the conservation of energy. We have

$$\partial_0 T^0{}_0 + \partial_i T^i{}_0 + \Gamma^\mu_{\mu 0} T^0{}_0 + \underbrace{\Gamma^\mu_{\mu i} T^i{}_0}_{O(2)} - \Gamma^0_{00} T^0{}_0 - \underbrace{\Gamma^0_{i0} T^i{}_0}_{O(2)} - \underbrace{\Gamma^i_{00} T^0{}_i}_{O(2)} - \Gamma^i_{j0} T^j{}_i = 0\,. \tag{3.3.76}$$

Substituting for the perturbed stress-energy tensor and the connection coefficients gives

$$\begin{aligned} \partial_\eta(\bar\rho + \delta\rho) + \partial_i q^i + (\mathcal{H} + \dot\psi + 3\mathcal{H} - 3\dot\phi)(\bar\rho + \delta\rho) - (\mathcal{H} + \dot\psi)(\bar\rho + \delta\rho) \\ -(\mathcal{H} - \dot\phi)\delta^i_j \left[-(\bar P + \delta P)\delta^j_i + \Pi^j{}_i\right] = 0 \\ \Rightarrow \quad \dot{\bar\rho} + \partial_\eta \delta\rho + \partial_i q^i + 3\mathcal{H}(\bar\rho + \delta\rho) - 3\bar\rho\dot\phi + 3\mathcal{H}(\bar P + \delta P) - 3\bar P\dot\phi = 0\,. \end{aligned} \tag{3.3.77}$$

The zero-order part gives the conservation of energy in the background,

$$\dot{\bar\rho} + 3\mathcal{H}(\bar\rho + \bar P) = 0\,, \tag{3.3.78}$$

and the first-order part gives

$$\partial_\eta \delta\rho + \partial_i q^i - 3(\bar\rho + \bar P)\dot\phi + 3\mathcal{H}(\delta\rho + \delta P) = 0\,. \tag{3.3.79}$$

The term $\partial_i q^i$ describes changes in energy in a volume due to flow through the surfaces; the remaining two terms describe the perturbation to the dilution of energy density by expansion (including PV work) with $-3\dot\phi$ arising from the conformal time derivative of the perturbation to the spatial volume element $(1 - 3\phi)d^3\mathbf{x}$. If we write $\delta\rho$ in terms of the fractional overdensity, $\delta\rho = \bar\rho\delta$, and q^i in terms of the peculiar velocity, $q^i = (\bar\rho + \bar P)v^i$, Eq. (3.3.79) becomes

$$\dot\delta + \left(1 + \frac{\bar P}{\bar\rho}\right)(\partial_i v^i - 3\dot\phi) + 3\mathcal{H}\left(\frac{\delta P}{\bar\rho} - \frac{\bar P}{\bar\rho}\delta\right) = 0\,. \tag{3.3.80}$$

In the limit $P \ll \rho$, we recover the Newtonian continuity equation in conformal time, $\dot\delta + \partial_i v^i - 3\dot\phi = 0$, but with a general-relativistic correction due to the perturbation to the rate of expansion of space.

If we now consider the ith component of Eq. (3.3.75), this will give the conservation of 3-momentum. We start with

$$\begin{aligned} \partial_\mu T^\mu{}_i + \Gamma^\mu_{\mu\rho} T^\rho{}_i - \Gamma^\rho{}_{\mu i} T^\mu{}_\rho &= 0 \\ \Rightarrow \quad \partial_\eta T^0{}_i + \partial_j T^j{}_i + \Gamma^\mu_{\mu 0} T^0{}_i + \Gamma^\mu_{\mu j} T^j{}_i - \Gamma^0_{0i} T^0{}_0 - \Gamma^0_{ji} T^j{}_0 - \Gamma^j_{0i} T^0{}_j - \Gamma^j_{ki} T^k{}_j &= 0\,. \end{aligned} \tag{3.3.81}$$

Note that this involves a mixed component of the stress-energy tensor, $T^0{}_i$ that we have not written down explicitly before – we can find it from Eqs (3.3.25) and (3.3.29) as follows:

$$\begin{aligned} T^0{}_i = g_{i\mu} T^{0\mu} &= g_{i0} T^{00} + g_{ij} T^{0j} \\ &= 0 - a^2(1-2\phi)\delta_{ij} a^{-2} q^i \\ &= -q_i\,. \end{aligned} \tag{3.3.82}$$

Equation (3.3.81) then becomes

$$\begin{aligned} -\dot{q}_i + \partial_j \left[-(\bar{P} + \delta P)\delta^j_i + \Pi^j{}_i \right] - 4\mathcal{H} q_i - (\partial_j \psi - 3\partial_j \phi)\bar{P}\delta^j_i - \partial_i \psi \bar{\rho} \\ -\mathcal{H}\delta_{ji} q^j + \mathcal{H}\delta^j_i q_j + \underbrace{\left(-2\delta^j_{(i}\partial_{k)}\phi + \delta_{ki}\delta^{jl}\partial_l \phi \right) \bar{P}\delta^k_j}_{-3\partial_i \phi \bar{P}} = 0 \\ \Rightarrow \quad -\dot{q}_i - \partial_i \delta P + \partial_j \Pi^j{}_i - 4\mathcal{H} q_i - (\bar{\rho} + \bar{P})\partial_i \psi = 0\,. \end{aligned} \tag{3.3.83}$$

Writing $q_i = (\bar{\rho} + \bar{P})v_i$, and using Eq. (3.3.78), we get the relativistic version of the Euler equation:

$$\dot{v}_i + \frac{1}{\bar{\rho} + \bar{P}} \partial_i \delta P - \frac{1}{\bar{\rho} + \bar{P}} \partial_j \Pi^j{}_i + \mathcal{H} v_i + \frac{\dot{\bar{P}}}{\bar{\rho} + \bar{P}} v_i + \partial_i \psi = 0\,. \tag{3.3.84}$$

This is like the Euler equation for a viscous fluid [c.f. Eq. (3.2.22)], with pressure gradients, the divergence of the anisotropic stress and gravitational infall driving $\dot{v}_i$, but with corrections due to redshifting of peculiar velocities ($\mathcal{H}v_i$) and P/ρ effects.

Once an equation of state of the matter (and other constitutive relations) are specified, we just need the gravitational potentials ψ and ϕ to close the system of equations of energy conservation.

Perturbed Einstein equation

We require the perturbation to the Einstein tensor, $G_{\mu\nu} \equiv R_{\mu\nu} - g_{\mu\nu}R/2$, so we first need to calculate the perturbed Ricci tensor $R_{\mu\nu}$ and scalar R. The Ricci tensor is a contraction of the Riemann tensor and can be expressed in terms of the connection as

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\rho} + \Gamma^\alpha_{\mu\nu}\Gamma^\rho_{\alpha\rho} - \Gamma^\alpha_{\mu\rho}\Gamma^\rho_{\nu\alpha}\,. \tag{3.3.85}$$

We shall derive the 00 component here with the others left as an exercise. We have

$$R_{00} = \partial_\rho \Gamma^\rho_{00} - \partial_\eta \Gamma^\rho_{0\rho} + \Gamma^\alpha_{00}\Gamma^\rho_{\alpha\rho} - \Gamma^\alpha_{0\rho}\Gamma^\rho_{0\alpha} \,. \tag{3.3.86}$$

When we sum over ρ, the terms with $\rho = 0$ cancel so we need only consider summing over $\rho = 1, 2, 3$, i.e.

$$\begin{aligned} R_{00} &= \partial_i \Gamma^i_{00} - \partial_\eta \Gamma^i_{0i} + \Gamma^\alpha_{00}\Gamma^i_{\alpha i} - \Gamma^\alpha_{0i}\Gamma^i_{0\alpha} \\ &= \partial_i \Gamma^i_{00} - \partial_\eta \Gamma^i_{0i} + \Gamma^0_{00}\Gamma^i_{0i} + \underbrace{\Gamma^j_{00}\Gamma^i_{ji}}_{O(2)} - \underbrace{\Gamma^0_{0i}\Gamma^i_{00}}_{O(2)} - \Gamma^j_{0i}\Gamma^i_{0j} \\ &= \nabla^2\psi - 3\partial_\eta(\mathcal{H} - \dot\phi) - 3(\mathcal{H} + \dot\psi)(\mathcal{H} - \dot\phi) - (\mathcal{H} - \dot\phi)^2 \delta^j_i \delta^i_j \\ &= -3\dot{\mathcal{H}} + \nabla^2\psi + 3\mathcal{H}(\dot\phi + \dot\psi) + 3\ddot\phi \,. \end{aligned} \tag{3.3.87}$$

Exercise: Show that the remaining components of the perturbed Ricci tensor are

$$R_{0i} = 2\partial_i\dot\phi + 2\mathcal{H}\partial_i\psi \tag{3.3.88}$$

$$R_{ij} = \left[\dot{\mathcal{H}} + 2\mathcal{H}^2 - \ddot\phi + \nabla^2\phi - 2(\dot{\mathcal{H}} + 2\mathcal{H}^2)(\phi + \psi) - \mathcal{H}\dot\psi - 5\mathcal{H}\dot\phi\right]\delta_{ij} + \partial_i\partial_j(\phi - \psi) \,. \tag{3.3.89}$$

We now form the Ricci tensor $g^{\mu\nu}R_{\mu\nu}$ using

$$R = g^{00}R_{00} + 2\underbrace{g^{0i}R_{0i}}_{0} + g^{ij}R_{ij} \,. \tag{3.3.90}$$

It follows that

$$\begin{aligned} a^2 R &= (1 - 2\psi)R_{00} - (1 + 2\phi)\delta^{ij}R_{ij} \\ &= (1 - 2\psi)\left[-3\dot{\mathcal{H}} + \nabla^2\psi + 3\mathcal{H}(\dot\phi + \dot\psi) + 3\ddot\phi\right] \\ &\quad - 3(1 + 2\phi)\left[\dot{\mathcal{H}} + 2\mathcal{H}^2 - \ddot\phi + \nabla^2\phi - 2(\dot{\mathcal{H}} + 2\mathcal{H}^2)(\phi + \psi) - \mathcal{H}\dot\psi - 5\mathcal{H}\dot\phi\right] \\ &\quad - (1 + 2\phi)\nabla^2(\phi - \psi) \,. \end{aligned} \tag{3.3.91}$$

Simplifying, we find, to linear order,

$$a^2 R = -6(\dot{\mathcal{H}} + \mathcal{H}^2) + 2\nabla^2\psi - 4\nabla^2\phi + 12(\dot{\mathcal{H}} + \mathcal{H}^2)\psi + 6\ddot\phi + 6\mathcal{H}(\dot\psi + 3\dot\phi) \,. \tag{3.3.92}$$

Finally, we can form the Einstein tensor. The 00 component is

$$\begin{aligned} G_{00} &= R_{00} - \frac{1}{2}g_{00}R \\ &= -3\dot{\mathcal{H}} + \nabla^2\psi + 3\mathcal{H}(\dot\phi + \dot\psi) + 3\ddot\phi + 3(1 + 2\psi)(\dot{\mathcal{H}} + \mathcal{H}^2) \\ &\quad - \frac{1}{2}\left[2\nabla^2\psi - 4\nabla^2\phi + 12(\dot{\mathcal{H}} + \mathcal{H}^2)\psi + 6\ddot\phi + 6\mathcal{H}(\dot\psi + 3\dot\phi)\right] \,. \end{aligned} \tag{3.3.93}$$

Most of the terms cancel leaving the simple result

$$G_{00} = 3\mathcal{H}^2 + 2\nabla^2\phi - 6\mathcal{H}\dot{\phi}\,. \tag{3.3.94}$$

The $0i$ component of the Einstein tensor is simply R_{0i} since $g_{0i} = 0$ in the conformal Newtonian gauge:

$$G_{0i} = 2\partial_i\dot{\phi} + 2\mathcal{H}\partial_i\psi\,. \tag{3.3.95}$$

The remaining components are

$$\begin{aligned} G_{ij} &= R_{ij} - \frac{1}{2}g_{ij}R \\ &= \left[\dot{\mathcal{H}} + 2\mathcal{H}^2 - \ddot{\phi} + \nabla^2\phi - 2(\dot{\mathcal{H}} + 2\mathcal{H}^2)(\phi + \psi) - \mathcal{H}\dot{\psi} - 5\mathcal{H}\dot{\phi}\right]\delta_{ij} + \partial_i\partial_j(\phi - \psi) \\ &\quad - 3(1 - 2\phi)(\dot{\mathcal{H}} + \mathcal{H}^2)\delta_{ij} \\ &\quad + \frac{1}{2}\left[2\nabla^2\psi - 4\nabla^2\phi + 12(\dot{\mathcal{H}} + \mathcal{H}^2)\psi + 6\ddot{\phi} + 6\mathcal{H}(\dot{\psi} + 3\dot{\phi})\right]\delta_{ij}\,. \end{aligned} \tag{3.3.96}$$

This neatens up (only a little!) to give

$$\begin{aligned} G_{ij} &= -(2\dot{\mathcal{H}} + \mathcal{H}^2)\delta_{ij} + \left[\nabla^2(\psi - \phi) + 2\ddot{\phi} + 2(2\dot{\mathcal{H}} + \mathcal{H}^2)(\phi + \psi) + 2\mathcal{H}\dot{\psi} + 4\mathcal{H}\dot{\phi}\right]\delta_{ij} \\ &\quad + \partial_i\partial_j(\phi - \psi)\,. \end{aligned} \tag{3.3.97}$$

Substituting the perturbed Einstein tensor, metric and stress-energy tensor into the Einstein equation gives the equations of motion for the metric perturbations and the zero-order Friedmann equations. For example,

$$\begin{aligned} G_{00} &= 8\pi G T_{00} + \Lambda g_{00} \\ \Rightarrow \quad 3\mathcal{H}^2 + 2\nabla^2\phi - 6\mathcal{H}\dot{\phi} &= 8\pi G\left(g_{00}T^0{}_0 + g_{0i}T^i{}_0\right) + \Lambda a^2(1 + 2\psi) \\ &= 8\pi G a^2\bar{\rho}(1 + 2\psi)(1 + \delta) + \Lambda a^2(1 + 2\psi)\,. \end{aligned} \tag{3.3.98}$$

The zero-order part gives

$$\mathcal{H}^2 = \frac{8\pi G}{3}a^2\bar{\rho} + \frac{1}{3}\Lambda a^2\,, \tag{3.3.99}$$

which is just the Friedmann equation (3.3.3). The first-order part of Eq. (3.3.98) gives

$$\nabla^2\phi = (8\pi G a^2\bar{\rho} + \Lambda a^2)\psi + 3\mathcal{H}\dot{\phi} + 4\pi G a^2\bar{\rho}\delta\,. \tag{3.3.100}$$

which, on using Eq. (3.3.99), reduces to

$$\nabla^2\phi = 3\mathcal{H}(\dot{\phi} + \mathcal{H}\psi) + 4\pi G a^2\bar{\rho}\delta\,. \tag{3.3.101}$$

The $0i$ Einstein equation is

$$G_{0i} = 8\pi G T_{0i} + \Lambda g_{0i}\,. \tag{3.3.102}$$

The last term on the right vanishes in the conformal Newtonian gauge, and

$$\begin{aligned} T_{0i} &= g_{0\mu}g_{i\nu}T^{\mu\nu} \\ &= g_{00}g_{ij}T^{0j} \quad (\text{since } g_{0i}=0) \\ &= -a^2(1+2\psi)a^2(1-2\phi)\delta_{ij}a^{-2}q^j \\ &= -a^2 q_i\,. \end{aligned} \tag{3.3.103}$$

It follows that

$$\partial_i\dot{\phi} + \mathcal{H}\partial_i\psi = -4\pi G a^2 q_i\,. \tag{3.3.104}$$

If we write $q_i = (\bar{\rho}+\bar{P})\partial_i v$ and assume the perturbations decay at infinity, we can integrate Eq. (3.3.104) to get

$$\dot{\phi} + \mathcal{H}\psi = -4\pi G a^2(\bar{\rho}+\bar{P})v\,. \tag{3.3.105}$$

Substituting this in the 00 Einstein equation gives

$$\nabla^2\phi = 4\pi G a^2\left[\bar{\rho}\delta - 3\mathcal{H}(\bar{\rho}+\bar{P})v\right]\,. \tag{3.3.106}$$

This is of the form of a Poisson equation but with source density $\bar{\rho}\delta - 3\mathcal{H}(\bar{\rho}+\bar{P})v$. What is the physical meaning of this term? It is numerically equal to the gauge-invariant variable $\bar{\rho}\Delta$ of Eq. (3.3.64) since $B=0$ in the conformal Newtonian gauge. Let us introduce *comoving hypersurfaces* as those that are orthogonal to the worldlines of a set of observers comoving with the total matter (i.e. they see $q^i=0$) and are the constant-time hypersurfaces in the *comoving gauge* for which $q^i=0$ and $B_i=0$. It follows that Δ is the fractional overdensity in the comoving gauge and we see from Eq. (3.3.106) that this is the source term for the gravitational potential ϕ.

The final content of the Einstein equation is contained in its ij components:

$$\begin{aligned} G_{ij} &= 8\pi G T_{ij} + \Lambda g_{ij} \\ &= 8\pi G g_{ik}g_{jl}T^{kl} - a^2\Lambda(1-2\phi)\delta_{ij} \\ &= 8\pi G a^4(1-2\phi)^2\delta_{ik}\delta_{jl}a^{-2}\left[\bar{P}\delta^{kl} + (2\bar{P}\phi+\delta P)\delta^{kl} - \Pi^{kl}\right] - a^2\Lambda(1-2\phi)\delta_{ij} \\ &= 8\pi G a^2(1-4\phi)\left[\bar{P}\delta_{ij} + (2\bar{P}\phi+\delta P)\delta_{ij} - \Pi_{ij}\right] - a^2\Lambda(1-2\phi)\delta_{ij} \\ &= a^2(8\pi G\bar{P}-\Lambda)\delta_{ij} + a^2\left[8\pi G(\delta P - 2\bar{P}\phi) + 2\Lambda\phi\right]\delta_{ij} - 8\pi G a^2\Pi_{ij}\,. \end{aligned} \tag{3.3.107}$$

Using Eq. (3.3.97) in the left-hand side, the zero-order part is

$$2\dot{\mathcal{H}} + \mathcal{H}^2 = -a^2(8\pi G\bar{P} - \Lambda)\,, \tag{3.3.108}$$

which, with Eq. (3.3.3) recover the second Friedmann equation (3.3.4). The first-order part of Eq. (3.3.107) is

$$\begin{aligned} &\left[\nabla^2(\psi-\phi) + 2\ddot{\phi} + 2(2\dot{\mathcal{H}}+\mathcal{H}^2)(\phi+\psi) + 2\mathcal{H}\dot{\psi} + 4\mathcal{H}\dot{\phi}\right]\delta_{ij} + \partial_i\partial_j(\phi-\psi) \\ &\qquad = a^2\left[8\pi G(\delta P - 2\bar{P}\phi) + 2\Lambda\phi\right]\delta_{ij} - 8\pi G a^2\Pi_{ij}\,. \end{aligned} \tag{3.3.109}$$

We can further manipulate this by considering the trace and trace-free parts separately. The latter is

$$\partial_{\langle i}\partial_{j\rangle}(\phi-\psi) = -8\pi G a^2 \Pi_{ij}\,. \tag{3.3.110}$$

This shows that, *in the absence of anisotropic stress* (and assuming appropriate decay at infinity), $\phi = \psi$ so there is then only one gauge-invariant degree of freedom in the metric. The remaining content of Eq. (3.3.109) is contained in the trace part; contracting with $\delta^{ij}/3$ gives

$$\begin{aligned}\nabla^2(\psi-\phi) + 2\ddot{\phi} + 2(2\dot{\mathcal{H}}+\mathcal{H}^2)(\phi+\psi) + 2\mathcal{H}\dot{\psi} + 4\mathcal{H}\dot{\phi} + \frac{1}{3}\nabla^2(\phi-\psi)& \\ = 8\pi G a^2 \delta P - 2a^2(8\pi G\bar{P} - \Lambda)\phi\,.&\end{aligned} \tag{3.3.111}$$

Rearranging, and using Eq. (3.3.108) in the right-hand side we get

$$\ddot{\phi} + \frac{1}{3}\nabla^2(\psi-\phi) + (2\dot{\mathcal{H}}+\mathcal{H}^2)\psi + \mathcal{H}\dot{\psi} + 2\mathcal{H}\dot{\phi} = 4\pi G a^2 \delta P\,. \tag{3.3.112}$$

Of course, the Einstein equations and the energy and momentum conservation equations form a redundant (but consistent!) set because of the Bianchi identity. We can use whichever subsets are most convenient for the particular problem at hand.

3.3.5 Density perturbations of a radiation fluid

We shall first consider the evolution of density perturbations in a universe containing only a radiation fluid, i.e. an ideal fluid with $P = \rho/3$. We shall later argue how this applies to the photons and neutrinos during radiation domination in our multi-component universe.

The basic idea is to get a closed equation for the potential ϕ (noting that $\phi = \psi$ since $\Pi_{ij} = 0$ by assumption) and then use this to determine the density fluctuations in the radiation. We start from Eq. (3.3.112) with $\delta P = \delta\rho/3$. In a universe dominated by radiation, $\mathcal{H}^2 \propto a^{-2}$ hence $a \propto \eta$ and $\mathcal{H} = 1/\eta$; it follows that

$$\ddot{\phi} + \frac{3}{\eta}\dot{\phi} - \frac{1}{\eta^2}\phi = \frac{4\pi G a^2\bar{\rho}}{3}\delta_r\,. \tag{3.3.113}$$

We can eliminate δ_r using the 00 Einstein equation in the form of Eq. (3.3.101) to find

$$\begin{aligned}&\ddot{\phi} + \frac{3}{\eta}\dot{\phi} - \frac{1}{\eta^2}\phi = \frac{1}{3}\nabla^2\phi - \frac{1}{\eta}\left(\dot{\phi} + \frac{1}{\eta}\phi\right) \\ \Rightarrow\quad &\ddot{\phi} + \frac{4}{\eta}\dot{\phi} - \frac{1}{3}\nabla^2\phi = 0\,.\end{aligned} \tag{3.3.114}$$

This is a damped wave equation with propagation speed $1/\sqrt{3}$ as expected for a radiation fluid where the square of the sound speed is $\partial P/\partial\rho = 1/3$. On Fourier expanding ($\nabla^2\phi \to -k^2\phi$), Eq. (3.3.114) gives

$$\ddot{\phi} + \frac{4}{\eta}\dot{\phi} + \frac{k^2}{3}\phi = 0\,. \tag{3.3.115}$$

Finally, writing $\phi = u(x)/x$ where $x \equiv k\eta/\sqrt{3}$, we have

$$u'' + \frac{2}{x}u' + \left(1 - \frac{2}{x^2}\right)u = 0\,. \tag{3.3.116}$$

This has independent solutions which are just the spherical Bessel functions $j_1(x)$ and $n_1(x)$[6]. These can be written explicitly as

$$\begin{aligned} j_1(x) &= \frac{\sin x}{x^2} - \frac{\cos x}{x} = \frac{x}{3} + O(x^3) \\ n_1(x) &= -\frac{\cos x}{x^2} - \frac{\sin x}{x} = -\frac{1}{x^2} + O(x^0)\,. \end{aligned} \tag{3.3.117}$$

The regular (growing-mode) solution for ϕ is thus

$$\phi \propto \frac{j_1(k\eta/\sqrt{3})}{k\eta}\,, \tag{3.3.118}$$

and is constant outside the *sound horizon*, i.e. for $k^{-1} \gg \eta/\sqrt{3}$. The asymptotic form of the spherical Bessel functions are

$$j_l(x) \sim \frac{1}{x}\sin\left(x - \frac{l\pi}{2}\right)\,, \tag{3.3.119}$$

hence $\phi \sim \cos(k\eta/\sqrt{3})/(k\eta)^2$ well inside the sound horizon. As expected from Eq. (3.3.115), we have oscillations at frequency $k/\sqrt{3}$ with a slow damping of the amplitude (over the order of a Hubble time).

The Poisson equation (3.3.106) relates the potential to the density perturbation in the comoving gauge. Here, this is the gauge that is comoving with the radiation fluid. Fourier expanding, we have

$$-k^2\phi = \frac{3}{2}\mathcal{H}^2\Delta_r = \frac{3}{2\eta^2}\Delta_r\,, \tag{3.3.120}$$

[6]Generally, spherical Bessel functions j_l and n_l satisfy

$$y_l'' + \frac{2}{x}y_l' + \left(1 - \frac{l(l+1)}{x^2}\right)y_l = 0\,,$$

where y_l is j_l or n_l.

so that

$$\Delta_r = -\frac{2}{3}(k\eta)^2\phi \propto k\eta j_1(k\eta/\sqrt{3})\,. \tag{3.3.121}$$

This grows like $\eta^2 \propto a^2$ outside the sound horizon and oscillates with constant amplitude inside the sound horizon. These sub-horizon oscillations are what give rise to the acoustic peaks in the power spectrum of the CMB anisotropies (see later).

We have to work a little harder to find the density perturbation in the conformal Newtonian gauge. Using Eq. (3.3.101), we have

$$\begin{aligned} -k^2\phi - \frac{3}{\eta}\left(\dot{\phi} + \frac{1}{\eta}\phi\right) &= \frac{3}{2\eta^2}\delta_r \\ \Rightarrow \quad \delta_r &= -\frac{2}{3}(k\eta)^2\phi - 2\eta\dot{\phi} - 2\phi\,. \end{aligned} \tag{3.3.122}$$

On large scales, $k\eta \ll 1$, ϕ is of the form $A + B(k\eta)^2$ so $\eta\dot{\phi} = 2B(k\eta)^2 \ll \phi$ and hence

$$\delta_r \approx -2\phi \quad (k\eta \ll 1) \tag{3.3.123}$$

and is constant. On small scales, $k\eta \gg 1$, ϕ oscillates and so $\dot{\phi} \sim k\phi$ and we find

$$\delta_r \approx -\frac{2}{3}(k\eta)^2\phi = \Delta_r\,. \tag{3.3.124}$$

We see that well inside the horizon, the density perturbations in the comoving and Newtonian gauge coincide. This is indicative of the general result that there are no gauge ambiguities inside the horizon.

Finally, from Eq. (3.3.104) we see that

$$v_i = -\frac{\eta^2}{2}\frac{\partial}{\partial x^i}\left(\dot{\phi} + \frac{1}{\eta}\phi\right)\,, \tag{3.3.125}$$

so the Newtonian-gauge peculiar velocity of the radiation fluid vanishes like $-k\eta\phi/2$ on large scales.

Perturbations in radiation during radiation domination

How does the discussion above apply to our real universe with its multiple components (e.g. baryons, CDM, radiation and neutrinos)? Well before matter-radiation equality (but well after nucleosynthesis), photons and neutrinos are the dominant components and the photons are kept isotropic in the rest frame of the baryons (here taken to include leptons, i.e. electrons) because scattering is very efficient. After neutrino decoupling, the neutrinos free-stream and their distribution function will generally not remain isotropic in the presence of fluctuations in their energy and momentum density and in the spacetime metric. However, neutrino anisotropic stress is produced causally – it requires a particle to free-stream of the order of the wavelength of the perturbation – so, for perturbations that are outside the Hubble radius, anisotropic stress can be ignored and $\phi = \psi$. On these large scales, the dynamical equations of the photons and the neutrinos are identical while the baryon density is much smaller than the photon energy density[7]. As we shall describe shortly, for an important class of initial conditions (called adiabatic) the photon and neutrino fractional overdensities and peculiar velocities are equal and, to the extent that neutrino anisotropic stress can be ignored, this condition is preserved in time since their dynamical equations are the same. On super-Hubble scales in radiation domination for adiabatic initial conditions, the photons and neutrinos effectively behave like a single *radiation fluid* whose fractional overdensity equals that of the photons (and neutrinos), $\delta_r = \delta_\gamma$, and is what we calculated above. On sub-Hubble scales in radiation domination, the above treatment is an approximate description of the fluctuations in the photons; a more careful treatment properly requires small corrections be made to account for differences in the clustering properties of the photons and free-streaming neutrinos.

Simple inflation models predict *initial* fluctuations that are adiabatic. This means that the energy densities of all species are constant on hypersurfaces for which the total energy density is constant, and all species have the same peculiar velocities. The statement that the peculiar velocities are equal is gauge invariant since $v_i \to v_i + \dot{L}_i$ under a gauge transformation and the $\dot{L}_i$ term cancels when differencing a pair of peculiar velocities. (A simpler way to see this is to note that relative velocities are observable; they do not require a choice of coordinates to define them.) For the fluctuations in the energy densities, the quantity

$$\frac{\delta\rho_I}{\bar{\rho}_I + \bar{P}_I} - \frac{\delta\rho_J}{\bar{\rho}_J + \bar{P}_J} \,, \tag{3.3.126}$$

where I and J label the species, is also gauge-invariant since $\delta\rho_I \to \delta\rho_I - T\dot{\bar{\rho}}_I$ and $\dot{\bar{\rho}}_I = -3\mathcal{H}(\bar{\rho}_I + \bar{P}_I)$. Moreover, it vanishes for adiabatic fluctuations since then all $\delta\rho_I = 0$ in a gauge for which the constant-time hypersurfaces coincide with those of uniform total density. It follows that for adiabatic fluctuations, the total density perturbation in any gauge,

$$\delta\rho_{\text{tot}} = \bar{\rho}_{\text{tot}}\delta_{\text{tot}} = \sum_I \bar{\rho}_I \delta_I \,, \tag{3.3.127}$$

is dominated by the species that is dominant in the background since all the δ_I are comparable; deep in the radiation era, this is the radiation. If we now consider the (total) comoving gauge, this is necessarily comoving with the radiation in the radiation era and the density perturbation in this gauge, which is the source of the gravitational

[7]This qualification of negligible baryon density is needed since the photons and baryons exchange momentum through scattering and this alters the evolution of their (common) peculiar velocity from that for a photon-only fluid if the baryons have comparable energy density to the photons.

potential via the Poisson equation (3.3.106), is just the perturbation to the radiation energy density on its comoving hypersurfaces, Δ_r.

We noted above that the adiabatic conditions are preserved between photons and neutrinos on super-Hubble scales. This actually holds true more generally, and in particular for mixtures of barotropic fluids [i.e. $P_I = P_I(\rho_I)$]. For the densities, this follows directly from Eq. (3.3.80) and the fact that relative velocities vanish initially (we shall show this explicitly in the next section for the case of radiation and CDM); for the velocities it follows from causality since velocities can only separate via causal processes (stress gradients here).

3.3.6 Evolution of CDM and baryon perturbations in radiation domination on large scales

We discussed the evolution of CDM perturbations on sub-Hubble scales using Newtonian theory in Section 3.2.5. However, to describe the evolution on large scales we require a relativistic treatment.

For adiabatic initial conditions, the CDM, baryon and radiation perturbations are related by

$$3\delta_r/4 = \delta_c = \delta_b \,, \tag{3.3.128}$$

and they co-move with each other. As advertised above, we can show that the condition Eq. (3.3.128) is preserved on large scales by considering their respective continuity equations. Equation (3.3.80) gives

$$\begin{aligned} \dot{\delta}_r + 4\partial_i v_r^i/3 - 4\dot{\phi} &= 0 \\ \dot{\delta}_c + \partial_i v_c^i - 3\dot{\phi} &= 0 \\ \Rightarrow \quad \partial_\eta \left(3\delta_r/4 - \delta_c\right) + \partial_i (v_r^i - v_c^i) &= 0 \,, \end{aligned} \tag{3.3.129}$$

and, since the relative velocities coincide on large scales, $3\delta_r/4$ remains equal to δ_c outside the horizon for adiabatic initial conditions. A similar treatment holds for the baryons and radiation, but now $v_r^i = v_b^i$ holds on all scales larger than the mean-free path to (Thomson) scattering for any initial condition. For adiabatic initial conditions, this implies $\delta_b = 3\delta_r/4$ on all scales where tight-coupling holds.

We see from this discussion that, for adiabatic initial conditions, $\delta_c = \delta_b = -3\phi/2$ on large scales in radiation domination and are constant. Once the mode enters the sound horizon, the radiation and baryons begin to oscillate acoustically and we can use the Newtonian treatment of the Meszaros effect in Section 3.2.5 to describe the CDM evolution.

3.3.7 Evolution of matter fluctuations in the matter era

In the matter era, well after recombination, the comoving frame moves with the matter (CDM and baryons). The total density perturbation on comoving hypersurfaces is then essentially Δ_m and this is related to ϕ via

$$\nabla^2\phi = 4\pi G a^2 \bar{\rho}_m \Delta_m \,. \tag{3.3.130}$$

The pressure fluctuation is negligible and so, from Eq. (3.3.112), the potential evolves as

$$\ddot{\phi} + 3\mathcal{H}\dot{\phi} + (2\dot{\mathcal{H}} + \mathcal{H}^2)\phi = 0 \,. \tag{3.3.131}$$

In matter domination, $a \propto \eta^2$ and $\mathcal{H} = 2/\eta$ so $2\dot{\mathcal{H}} + \mathcal{H}^2 = 0$ and we find

$$\ddot{\phi} + \frac{6}{\eta}\dot{\phi} = 0 \,. \tag{3.3.132}$$

The solutions of this are $\phi = \text{const.}$ and a decaying solution $\phi \propto \eta^{-5} \propto a^{-5/2}$, just as in Newtonian theory (see equation 3.2.35) but now valid on all scales. The comoving density contrast therefore evolves as

$$\Delta_m \propto a^{-2} \underbrace{a^3}_{\text{from } \bar{\rho}} \underbrace{\begin{pmatrix} 1 \\ a^{-5/2} \end{pmatrix}}_{\text{from } \phi} \propto \begin{pmatrix} a \\ a^{-3/2} \end{pmatrix} . \tag{3.3.133}$$

These also agree with the Newtonian treatment, but note that this is for the comoving density contrast and the Newtonian-gauge result differs (the growing mode is constant) outside the horizon.

More generally, the late-time evolution of Δ_m follows from combining Eqs (3.3.130) and (3.3.131). Since $a^2\bar{\rho}_m \propto a^{-1}$, we have

$$\partial_\eta^2(\Delta_m/a) + 3\mathcal{H}\partial_\eta(\Delta/a) + (2\dot{\mathcal{H}} + \mathcal{H}^2)(\Delta_m/a) = 0 \,, \tag{3.3.134}$$

which rearranges to (exercise!)

$$\ddot{\Delta}_m + \mathcal{H}\dot{\Delta}_m + (\dot{\mathcal{H}} - \mathcal{H}^2)\Delta_m = 0 \,. \tag{3.3.135}$$

The Friedmann equations (3.3.3) and (3.3.4) for matter and Λ give

$$\begin{aligned} \dot{\mathcal{H}} - \mathcal{H}^2 &= \frac{1}{3}a^2\Lambda - \frac{4\pi G a^2}{3}\bar{\rho}_m - \frac{1}{3}a^2\Lambda - \frac{8\pi G a^2}{3}\bar{\rho}_m \\ &= -4\pi G a^2 \bar{\rho}_m \,, \end{aligned} \tag{3.3.136}$$

so, generally, the comoving density contrast evolves as

$$\ddot{\Delta}_m + \mathcal{H}\dot{\Delta}_m - 4\pi G a^2 \bar{\rho}_m \Delta_m = 0 \,. \tag{3.3.137}$$

This is just the conformal-time version of the Newtonian equation (3.2.40) and so we recover the usual suppression of the growth of structure by Λ but now on all scales.

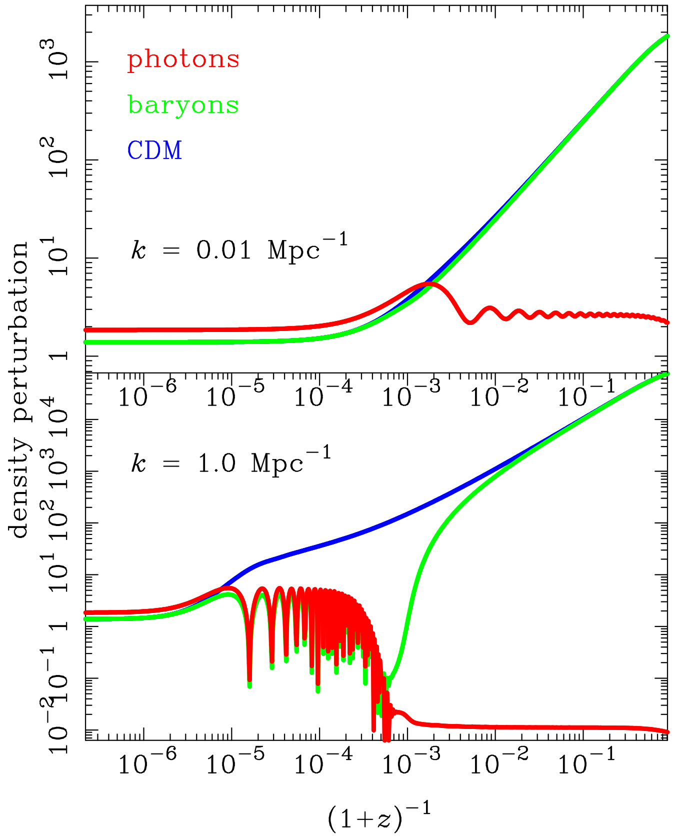

Figure 4: The variation of the density contrast in photons (red), baryons (green) and CDM (blue) with scale factor for $k = 0.01\,\mathrm{Mpc}^{-1}$ (top) and $k = 1.0\,\mathrm{Mpc}^{-1}$ (bottom). The density contrasts are plotted in the conformal Newtonian gauge.

3.3.8 Summary of growth of large-scale structure

The evolution of the perturbations to the energy densities of photons, baryons and CDM in the Newtonian gauge are illustrated in Fig. 4 for adiabatic initial conditions. The main points to note are as follows.

- The evolution is adiabatic (i.e. $\delta\rho_I/(\bar{\rho}_I + \bar{P}_I) = \delta\rho_J/(\bar{\rho}_J + \bar{P}_J)$ for all species I and J) until modes enter the Hubble radius. The Newtonian-gauge densities are constant outside the horizon, but the comoving gauge densities grow.
- Radiation and baryons undergo acoustic oscillations for modes that enter the sound horizon prior to decoupling.
- If modes additionally enter the Hubble radius during radiation domination, the CDM perturbation only grows logarithmically (Meszaros effect) until matter-radiation equality, after which it grows like a.

- Baryons quickly fall into the CDM potential wells after decoupling and $\delta_b \to \delta_c$.
- Once dark energy comes to dominate the expansion, the density perturbation growth slows down and eventually the perturbations freeze out.

3.3.9 Comoving curvature perturbation

There is an important quantity that is conserved on super-Hubble scales for adiabatic, scalar fluctuations irrespective of the equation of state of the matter: the *comoving curvature perturbation*. This is the perturbation to the intrinsic curvature scalar of comoving hypersurfaces, i.e. those hypersurfaces orthogonal to the worldlines that co-move with the total matter (i.e. for which $q^i = 0$). Its importance is that it allows us to match the perturbations from inflation to those in the radiation-dominated universe on large scales without needing to know the (uncertain) details of the reheating phase at the end of inflation.

In some arbitrary gauge, let us work out the *intrinsic curvature* of surfaces of constant time. The *induced metric*, γ_{ij}, on these surfaces is just the spatial part of Eq. (3.3.5), i.e.

$$\gamma_{ij} \equiv a^2 \left[(1-2\phi)\delta_{ij} + 2E_{ij}\right] , \tag{3.3.138}$$

and is not Euclidean because of the perturbations ϕ and E_{ij}. The 3D metric has an associated (metric-compatible) connection

$${}^{(3)}\Gamma^i_{jk} = \frac{1}{2}\gamma^{il}\left(\partial_j \gamma_{kl} + \partial_k \gamma_{jl} - \partial_l \gamma_{jk}\right) , \tag{3.3.139}$$

where γ^{ij} is the inverse of the induced metric,

$$\gamma^{ij} = a^{-2}\left[(1+2\phi)\delta^{ij} - 2E^{ij}\right] . \tag{3.3.140}$$

We actually only need the inverse metric to zero-order to compute the connection to first-order since the spatial derivatives of the γ_{ij} are all first-order in the perturbations. We have

$$\begin{aligned} {}^{(3)}\Gamma^i_{jk} &= \delta^{il}\partial_j\left(-\phi\delta_{kl} + E_{kl}\right) + \delta^{il}\partial_k\left(-\phi\delta_{jl} + E_{jl}\right) - \delta^{il}\partial_l\left(-\phi\delta_{jk} + E_{jk}\right) \\ &= -\left(2\delta^i_{(j}\partial_{k)}\phi - \delta^{il}\delta_{jk}\partial_l\phi\right) + \left(2\partial_{(j}E_{k)}{}^i - \delta^{il}\partial_l E_{jk}\right) . \end{aligned} \tag{3.3.141}$$

We can form the intrinsic curvature tensor in the same way as in 4D spacetime. The intrinsic curvature is the associated Ricci scalar, given by

$${}^{(3)}R = \gamma^{ik}\partial_l {}^{(3)}\Gamma^l_{ik} - \gamma^{ik}\partial_k {}^{(3)}\Gamma^l_{il} + \gamma^{ik(3)}\Gamma^l_{ik}{}^{(3)}\Gamma^m_{lm} - \gamma^{ik(3)}\Gamma^{m(3)}_{il}\Gamma^l_{km} . \tag{3.3.142}$$

To first-order, we therefore have

$$a^{2(3)}R = \delta^{ik}\partial_l {}^{(3)}\Gamma^l_{ik} - \delta^{ik}\partial_k {}^{(3)}\Gamma^l_{il} . \tag{3.3.143}$$

This involves two contractions of the connection. The first is

$$\begin{aligned}\delta^{ik\,(3)}\Gamma^l_{ik} &= -\delta^{ik}\left(2\delta^l_{(i}\partial_{k)}\phi - \delta^{jl}\delta_{ik}\partial_j\phi\right) + \delta^{ik}\left(2\partial_{(i}E_{k)}{}^l - \delta^{jl}\partial_j E_{ik}\right)\\ &= -2\delta^{kl}\partial_k\phi + 3\delta^{jl}\partial_j\phi + 2\partial_i E^{il} - \delta^{jl}\partial_j\underbrace{\left(\delta^{ik}E_{ik}\right)}_{0}\\ &= \delta^{kl}\partial_k\phi + 2\partial_k E^{kl}\,. \end{aligned} \tag{3.3.144}$$

The second is

$$\begin{aligned}{}^{(3)}\Gamma^l_{il} &= -\delta^l_l\partial_i\phi - \delta^l_i\partial_l\phi + \partial_i\phi + \partial_l E_i{}^l + \partial_i E_l{}^l - \partial_l E_i{}^l\\ &= -3\partial_i\phi\,. \end{aligned} \tag{3.3.145}$$

Using these results in Eq. (3.3.143) gives

$$\begin{aligned}a^{2\,(3)}R &= \partial_l\left(\delta^{kl}\partial_k\phi + 2\partial_k E^{kl}\right) + 3\delta^{ik}\partial_k\partial_i\phi\\ &= \nabla^2\phi + 2\partial_i\partial_j E^{ij} + 3\nabla^2\phi\\ &= 4\nabla^2\phi + 2\partial_i\partial_j E^{ij}\,. \end{aligned} \tag{3.3.146}$$

Note that this vanishes for vector and tensor perturbations (as do all perturbed scalars) since then $\phi = 0$ and $\partial_i\partial_j E^{ij} = 0$. For scalar perturbations, $E_{ij} = \partial_{\langle i}\partial_{j\rangle}E$ so

$$\begin{aligned}\partial_i\partial_j E^{ij} &= \delta^{il}\delta^{jm}\partial_i\partial_j\left(\partial_l\partial_m E - \frac{1}{3}\delta_{lm}\nabla^2 E\right)\\ &= \nabla^2\nabla^2 E - \frac{1}{3}\nabla^2\nabla^2 E = \frac{2}{3}\nabla^4 E,. \end{aligned} \tag{3.3.147}$$

Finally, we find

$$a^{2\,(3)}R = 4\nabla^2\left(\phi + \frac{1}{3}\nabla^2 E\right)\,. \tag{3.3.148}$$

We define the *curvature perturbation* as $-(\phi + \nabla^2 E/3)$ and the *comoving curvature perturbation* $\mathcal{R}$ is this quantity evaluated in the comoving gauge ($B_i = 0 = q^i$). It will prove convenient to have a gauge-invariant expression for $\mathcal{R}$ so that we can evaluate it from the perturbations in any gauge. Since B and v vanish in the comoving gauge, we can always add linear combinations of these to $-(\phi+\nabla^2 E/3)$ to form a gauge-invariant combination that equals $\mathcal{R}$.

Exercise: Use Eqs (3.3.55) and (3.3.59)–(3.3.61), to show that

$$\mathcal{R} = -\phi - \frac{1}{3}\nabla^2 E + \mathcal{H}(B + v) \tag{3.3.149}$$

is a gauge-invariant expression for the comoving curvature perturbation.

We now want to prove that $\mathcal{R}$ is conserved on large scales for adiabatic perturbations. We shall do so by working in the conformal Newtonian gauge, in which case $\mathcal{R} = -\phi + \mathcal{H}v$ (since B and E vanish). We can now use the $0i$ Einstein equation (3.3.105) to eliminate the peculiar velocity in favour of the potentials:

$$\mathcal{R} = -\phi - \frac{\mathcal{H}(\dot\phi + \mathcal{H}\psi)}{4\pi G a^2(\bar\rho + \bar P)} . \tag{3.3.150}$$

We now differentiate to find

$$\begin{aligned}
-4\pi G a^2(\bar\rho+\bar P)\dot{\mathcal{R}} &= 4\pi G a^2(\bar\rho+\bar P)\dot\phi + \dot{\mathcal{H}}(\dot\phi+\mathcal{H}\psi) + \mathcal{H}(\ddot\phi + \dot{\mathcal{H}}\psi + \mathcal{H}\dot\psi) \\
&\quad - \frac{\mathcal{H}(\dot\phi+\mathcal{H}\psi)}{(\bar\rho+\bar P)}\left(2\mathcal{H}(\bar\rho+\bar P) + \dot{\bar\rho} + \dot{\bar P}\right) \\
&= 4\pi G a^2(\bar\rho+\bar P)\dot\phi + \dot{\mathcal{H}}(\dot\phi+\mathcal{H}\psi) + \mathcal{H}(\ddot\phi + \dot{\mathcal{H}}\psi + \mathcal{H}\dot\psi) \\
&\quad - \mathcal{H}(\dot\phi+\mathcal{H}\psi)\left[2\mathcal{H} - 3\mathcal{H}\left(1+\frac{\dot{\bar P}}{\dot{\bar\rho}}\right)\right] ,
\end{aligned} \tag{3.3.151}$$

where we used $\dot{\bar\rho} = -3\mathcal{H}(\bar\rho+\bar P)$ in the second equality. In the term involving $\dot{\bar P}$, we now use the Poisson equation in the form of Eq. (3.3.101) to replace $\dot\phi + \mathcal{H}\psi$ with a combination of the potential and the density contrast, and in the first term on the right we use $\mathcal{H}^2 - \dot{\mathcal{H}} = 4\pi G a^2(\bar\rho + \bar P)$ to find

$$\begin{aligned}
-4\pi G a^2(\bar\rho+\bar P)\dot{\mathcal{R}} &= (\mathcal{H}^2 - \dot{\mathcal{H}})\dot\phi + \dot{\mathcal{H}}(\dot\phi+\mathcal{H}\psi) + \mathcal{H}(\ddot\phi + \dot{\mathcal{H}}\psi + \mathcal{H}\dot\psi) \\
&\quad + \mathcal{H}^2(\dot\phi+\mathcal{H}\psi) + \frac{\mathcal{H}\dot{\bar P}}{\dot{\bar\rho}}\left(\nabla^2\phi - 4\pi G a^2\bar\rho\delta\right) .
\end{aligned} \tag{3.3.152}$$

Adding and subtracting $4\pi G a^2\mathcal{H}\delta P$ on the right-hand side and simplifying gives

$$\begin{aligned}
-4\pi G a^2(\bar\rho+\bar P)\dot{\mathcal{R}} &= \mathcal{H}\left[\ddot\phi + \mathcal{H}\dot\psi + 2\mathcal{H}\dot\phi + (2\dot{\mathcal{H}} + \mathcal{H}^2)\psi - 4\pi G a^2\delta P\right] \\
&\quad + 4\pi G a^2\mathcal{H}\underbrace{\left(\delta P - \frac{\dot{\bar P}}{\dot{\bar\rho}}\bar\rho\delta\right)}_{\delta P_{\rm nad}} + \frac{\mathcal{H}\dot{\bar P}}{\dot{\bar\rho}}\nabla^2\phi .
\end{aligned} \tag{3.3.153}$$

Here, we have introduced the *non-adiabatic pressure perturbation*. It is gauge-invariant, since $\delta P \to \delta P - T\dot{\bar P}$ and $\delta\rho \to \delta\rho - T\dot{\bar\rho}$, and it vanishes for a barotropic equation of state, $P = P(\rho)$ such as in a fluid with no entropy perturbations. More generally, it vanishes for adiabatic fluctuations (see Section 3.3.5) in a mixture of barotropic fluids (since then, there exist hypersurfaces on which all the $\delta\rho_I$ vanish, and, if all species are barotropic, the δP_I will also vanish on these hypersurfaces). Finally, if we use the trace-part of the ij Einstein equation (3.3.112) we find

$$-4\pi G a^2(\bar\rho+\bar P)\dot{\mathcal{R}} = \frac{1}{3}\mathcal{H}\nabla^2(\phi-\psi) + 4\pi G a^2\mathcal{H}\delta P_{\rm nad} + \frac{\mathcal{H}\dot{\bar P}}{\dot{\bar\rho}}\nabla^2\phi . \tag{3.3.154}$$

If the non-adiabatic pressure vanishes, the right-hand side $\sim \mathcal{H}k^2\phi \sim \mathcal{H}k^2\mathcal{R}$ at scale k (where we have used equation 3.3.150), so that $\dot{\mathcal{R}}/\mathcal{H} \sim (k/\mathcal{H})^2\mathcal{R}$ and vanishes on super-Hubble scales[8].

We saw earlier that for adiabatic fluctuations the potential is constant during radiation domination on scales larger than the sound horizon, and on all (linear) scales during matter domination. How does the potential change during the matter-radiation transition on large scales? During radiation domination, a constant potential is related to the (super-Hubble) $\mathcal{R}$ by

$$\mathcal{R} = -\phi - \frac{\mathcal{H}^2\phi}{16\pi G\bar{\rho}a^2/3} = -\phi - \frac{\mathcal{H}^2\phi}{2\mathcal{H}^2} = -\frac{3}{2}\phi\,. \tag{3.3.155}$$

During matter domination, we have instead

$$\mathcal{R} = -\phi - \frac{\mathcal{H}^2\phi}{4\pi G\bar{\rho}a^2} = -\phi - \frac{\mathcal{H}^2\phi}{3\mathcal{H}^2/2} = -\frac{5}{3}\phi\,. \tag{3.3.156}$$

Since $\mathcal{R}$ is constant, we have

$$\frac{5}{3}\phi|_{\text{matter}} = \frac{3}{2}\phi|_{\text{radiation}} \quad\Rightarrow\quad \phi|_{\text{matter}} = \frac{9}{10}\phi|_{\text{radiation}} = -\frac{3}{5}\mathcal{R}\,. \tag{3.3.157}$$

This result is useful for relating the amplitude of the large-angle fluctuations in the CMB to the amplitude of primordial fluctuations.

In Section 3.5, we shall compute the statistics of $\mathcal{R}$ at the end of inflation in Fourier space. These statistics are related to those of any (linear) observable, since, in linear perturbation theory, different scales decouple so the Fourier transform of some observable, such as $\delta(\mathbf{k})$, will be linear in the primordial $\mathcal{R}(\mathbf{k})$.

3.4 Cosmic microwave background anisotropies

The cosmic microwave background (CMB) radiation is an almost perfect blackbody with temperature 2.725 K. If we look a little deeper, the first structure that is seen is a dipolar angular variation in the temperature which can be attributed to our motion relative to the rest-frame of the CMB. Consider us moving relative to the CMB with 3-velocity $\mathbf{v}$. Then a photon that we detect at energy E moving along direction $\mathbf{e}$ has 4-momentum $p^\mu = E(1, \mathbf{e})$. The energy of the photon in the CMB frame is $E_{\text{CMB}} = p_\mu u^\mu_{\text{CMB}}$ where $u^\mu_{\text{CMB}} = \gamma(1, -\mathbf{v})$ is the 4-velocity of the CMB rest-frame (i.e. the frame in which there is no photon momentum density hence dipole). We thus have

$$E_{\text{CMB}} = E\gamma(1 + \mathbf{v}\cdot\mathbf{e})\,. \tag{3.4.1}$$

[8]Note that the first term on the right-hand side of Eq. (3.3.154) vanishes on all scales if the anisotropic stress vanishes.

The photons are described by a (Lorentz-invariant) distribution function $f(p^\mu) \propto 1/(e^{E_{\mathrm{CMB}}/T_{\mathrm{CMB}}} - 1)$ which, in terms of the energy and direction seen by us, is

$$f(p^\mu) \propto \frac{1}{e^{E\gamma(1+\mathbf{v}\cdot\mathbf{e})/T_{\mathrm{CMB}}} - 1} \,. \tag{3.4.2}$$

This still looks like a blackbody along any direction but the observed temperature varies over the sky as

$$T(\mathbf{e}) = \frac{T_{\mathrm{CMB}}}{\gamma(1+\mathbf{v}\cdot\mathbf{e})} \approx T_{\mathrm{CMB}}(1 - \mathbf{v}\cdot\mathbf{e}) \tag{3.4.3}$$

for $|\mathbf{v}| \ll 1$. This is a dipole anisotropy. The measured dipole implies the solar system barycentre has speed $3.68 \times 10^5\,\mathrm{m\,s^{-1}}$ relative to the CMB. It is clear from Eq. (3.4.3) that relative motion also produces quadrupole anisotropies and alters the monopole temperature but only at $O(|\mathbf{v}|^2)$.

3.4.1 Temperature anisotropies from scalar perturbations

Metric fluctuations lead to changes in a photon's energy during propagation relative to the change in an unperturbed cosmology. This effect, in turn, leads to CMB temperature fluctuations on the sky.

We shall do the calculation of the effect of metric fluctuations on the CMB in the conformal Newtonian gauge of Section 3.3.4. It is convenient to parameterise the photon 4-momentum in terms of the energy E seen by an observer at rest in the coordinates and by the direction cosines e^i of the propagation direction seen by the same observer on the $(E_i)^\mu$ orthonormal frame of vectors (see Eq. 3.3.21). Note that $\delta_{ij}e^i e^j = 1$. We then have

$$\begin{aligned} p^\mu &= E[(E_0)^\mu + e^i (E_i)^\mu] \\ &= E a^{-1}[(1-\psi)\delta^\mu_0 + e^i(1+\phi)\delta^\mu_i] \,. \end{aligned} \tag{3.4.4}$$

Using the obvious 3-vector notation, whereby we identify e^i with the 3-vector $\mathbf{e}$, we can write

$$p^\mu = \underbrace{Ea^{-1}}_{\epsilon a^{-2}}[(1-\psi), (1+\phi)\mathbf{e}] \,, \tag{3.4.5}$$

where we have introduced the *comoving energy* $\epsilon \equiv Ea$ which is constant in the background.

Photons move on geodesics of the perturbed metric so

$$\frac{dp^\mu}{d\lambda} + \Gamma^\mu_{\nu\rho} p^\nu p^\rho = 0 \,, \tag{3.4.6}$$

where λ is an *affine parameter* such that $p^\mu = dx^\mu/d\lambda$. Using the parameterisation of Eq. (3.4.5) in $p^\mu = dx^\mu/d\lambda$, we have

$$\begin{aligned} \frac{d\eta}{d\lambda} &= \frac{\epsilon}{a^2}(1-\psi) \\ \frac{dx^i}{d\lambda} &= \frac{\epsilon}{a^2}(1+\phi)e^i \\ \Rightarrow \quad \frac{dx^i}{d\eta} &= (1+\phi+\psi)e^i \end{aligned} \tag{3.4.7}$$

at linear order. The geodesic equation in conformal time is

$$(1-\psi)\frac{\epsilon}{a^2}\frac{dp^\mu}{d\eta} + \Gamma^\mu_{\nu\rho}p^\nu p^\rho = 0\,. \tag{3.4.8}$$

The 0-component of this is

$$\begin{aligned} (1-\psi)\frac{\epsilon}{a^2}\frac{d}{d\eta}\left(\frac{\epsilon}{a^2}(1-\psi)\right) + \frac{\epsilon^2}{a^4}\left[\Gamma^0_{00}(1-2\psi) + 2\Gamma^0_{0i}e^i + \Gamma^0_{ij}(1+2\phi)e^ie^j\right] = 0 \\ \Rightarrow \quad \frac{1}{\epsilon}\frac{d\epsilon}{d\eta} - 2\mathcal{H} - \frac{d\psi}{d\eta} + \mathcal{H} + \dot{\psi} + 2e^i\partial_i\psi \\ +\delta_{ij}e^ie^j\left[\mathcal{H} - \dot{\phi} - 2\mathcal{H}(\phi+\psi)\right]\left[1+2(\phi+\psi)\right] = 0\,, \end{aligned} \tag{3.4.9}$$

where we have substituted for the perturbed connection coefficients from Eqs (3.3.68)–(3.3.73). The derivative $d\psi/d\eta$ is along the path of the photon so

$$d\psi/d\eta = \dot{\eta} + e^i\partial_i\psi \tag{3.4.10}$$

to first order. Adding and subtracting twice this term to Eq. (3.4.9) and simplifying gives

$$\frac{1}{\epsilon}\frac{d\epsilon}{d\eta} = -\frac{d\psi}{d\eta} + (\dot{\phi} + \dot{\psi})\,. \tag{3.4.11}$$

This equation tells us how the comoving energy evolves along the photon path in the presence of the metric perturbations. In the background, ϵ is constant, but this is modified by variation of ψ along the path (the first term on the right) and by time evolution of the potentials (second term on the right). For the CMB, the latter is important at late times, once dark energy dominates, and also around the time of last scattering due to the universe not being fully matter dominated at that time.

The ith component of the geodesic equation is left as an exercise.

Exercise: Show that the direction of propagation evolves along the photon path according to the first-order equation

$$\frac{de^i}{d\eta} = -(\delta^{ij} - e^ie^j)\partial_j(\phi+\psi)\,. \tag{3.4.12}$$

Equation (3.4.12) describes *gravitational lensing* whereby the photon direction is disturbed by the gradient of the potential $\phi + \psi$ perpendicular to the line of sight. In the background model, the direction is constant and this is all we require to compute the CMB temperature anisotropies since the dependence of the temperature on direction is already first order.

On large scales ($> 30\,\mathrm{Mpc}$) we can think of the CMB photons as being released instantaneously at the *time of last scattering.* (The probability density of a photon last scattering at some time η is called the *visibility function.* It peaks sharply at some η_* corresponding to $z = 1088$ in the concordance ΛCDM model with a width of a few tens of Mpc. On scales larger than that, the visibility can be approximated as a delta function centred on the peak of the visibility.) In the perturbed universe, a useful approximation to make[9] is that last scattering occurs on a hypersurface of constant photon energy density equal to the density in the background at η_*, $\bar{\rho}_\gamma(\eta_*)$. Over this surface, the photon distribution is isotropic in the rest frame of the baryons due to tight coupling (up to that point) and is a blackbody with temperature $\bar{T}_*$.

We now integrate Eq. (3.4.11) from the point of emission on the perturbed last scattering surface, E, to the observation point, 0:

$$\begin{aligned} \ln\left(\frac{\epsilon_0}{\epsilon_E}\right) &= -(\psi_0 - \psi_E) + \int_E^0 (\dot{\phi} + \dot{\psi})\, d\eta \\ \Rightarrow \quad \frac{\epsilon_0}{\epsilon_E} &= 1 + (\psi_E - \psi_0) + \int_E^0 (\dot{\phi} + \dot{\psi})\, d\eta \end{aligned} \tag{3.4.13}$$

to first order. The comoving energy of the photon thus changes due to changes in the gravitational potential ψ between the emission and reception events (gravitational redshifting) and also from evolution of the potentials in time. For the former, if the potential well is deeper at emission than reception $\psi_E < \psi_0$, there is a net loss of energy (a redshift). For the latter, for decaying potentials (such as when dark energy dominates) traversing an overdense region gives a net increase in energy (blueshift) since the photon gains more energy falling into the well (when it is deeper) than climbing out. The comoving energy of the emitted photon in the baryon rest frame at last scattering is

$$\epsilon_b = \epsilon_E(1 - e_i v_b^i)\,, \tag{3.4.14}$$

where we have used the Doppler formula, Eq. (3.4.1), with $\mathbf{v} = -\mathbf{v}_b$. It follows that there is a further Doppler shift in the received energy when referenced to an observer comoving with the baryons at last scattering:

$$\frac{\epsilon_0}{\epsilon_b} = 1 + (\psi_E - \psi_0) + e_i v_b^i + \int_E^0 (\dot{\phi} + \dot{\psi})\, d\eta\,. \tag{3.4.15}$$

[9] This can be avoided by a full kinetic theory treatment of anisotropy generation where one deals with the visibility function directly; see Dodelson, *Modern Cosmology* or next term's *Advanced Cosmology* course for details.

A photon that last scatters of an electron moving towards us (i.e. along $\mathbf{e}$) will appear more energetic when received than one scattered off an electron moving away from us.

The ratio of the temperature of the CMB received by the observer at rest in the Newtonian gauge, $T(\mathbf{e})$, to the temperature of the photons on the last-scattering surface, $\bar{T}_*$, is just the ratio of the proper energies of the photon at reception and emission (in the baryon frame):

$$\frac{T(\mathbf{e})}{\bar{T}_*} = \frac{\epsilon_0}{\epsilon_b}\frac{a_E}{a_0}\,, \tag{3.4.16}$$

where a_E is the scale factor at the (perturbed) point of emission. It follows that

$$\frac{T(\mathbf{e})}{\bar{T}_*} = \frac{a_E}{a_0}\left(1 + (\psi_E - \psi_0) + e_i v_b^i + \int_E^0 (\dot{\phi} + \dot{\psi})\, d\eta\right)\,. \tag{3.4.17}$$

The final issue is to determine how a_E varies over the perturbed last-scattering surface. For a photon observed at 0 (which we take to be the origin of our spatial coordinates) directed along $\mathbf{e}$, it will have last scattered at an event with spatial coordinates $-(\eta_0 - \eta_*)e^i + \Delta x^i(\mathbf{e})$ and time coordinate $\eta_* + \Delta\eta(\mathbf{e})$, where $\Delta x^i(\mathbf{e})$ and $\Delta\eta(\mathbf{e})$ are small perturbations from the emission event in the background. The photon energy density at the event E in terms of the Newtonian-gauge density contrast is

$$\bar{\rho}_\gamma(\eta_* + \Delta\eta)(1 + \delta_\gamma) = \bar{\rho}_\gamma(\eta_*)\,, \tag{3.4.18}$$

where we have used the fact that the energy density is uniform over the true last-scattering surface and equal to the background value at last scattering. Note that δ_γ can be evaluated at the background position to the required (first) order. Taylor expanding in Eq. (3.4.18), we have

$$\begin{aligned} \bar{\rho}_\gamma(\eta_*) &= \bar{\rho}_\gamma(\eta_*)\left(1 + \Delta\eta(\mathbf{e})\frac{\dot{\bar{\rho}}_\gamma}{\bar{\rho}_\gamma} + \delta_\gamma\right) \\ \Rightarrow \quad \Delta\eta(\mathbf{e}) &= -\frac{\delta_\gamma}{\dot{\bar{\rho}}_\gamma/\bar{\rho}_\gamma} = \frac{\delta_\gamma}{4\mathcal{H}}\,. \end{aligned} \tag{3.4.19}$$

We thus have

$$a_E = a[\eta_* + \Delta\eta(\mathbf{e})] = a(\eta_*)[1 + \mathcal{H}\Delta\eta(\mathbf{e})] = a_*(1 + \delta_\gamma/4)\,, \tag{3.4.20}$$

and so

$$\begin{aligned} \frac{T(\mathbf{e})}{\bar{T}_*} &= \frac{a_*}{a_0}\left(1 + \frac{1}{4}\delta_\gamma + (\psi_E - \psi_0) + e_i v_b^i + \int_E^0 (\dot{\phi} + \dot{\psi})\, d\eta\right) \\ \Rightarrow \quad T(\mathbf{e}) &= \bar{T}_0\left(1 + \frac{1}{4}\delta_\gamma + (\psi_E - \psi_0) + e_i v_b^i + \int_E^0 (\dot{\phi} + \dot{\psi})\, d\eta\right)\,, \end{aligned} \tag{3.4.21}$$

where $\bar{T}_0 \equiv \bar{T}_* a_*/a_0$ is the background CMB temperature at the point of reception. We thus define the fractional temperature perturbation as

$$\frac{\Delta T(\mathbf{e})}{\bar{T}_0} \equiv \frac{T(\mathbf{e}) - \bar{T}_0}{\bar{T}_0} = \frac{1}{4}\delta_\gamma + (\psi_E - \psi_0) + e_i v_b^i + \int_E^0 (\dot{\phi} + \dot{\psi})\, d\eta \tag{3.4.22}$$

Note that the potential at the reception point, ψ_0, only affects the monopole perturbation of the CMB temperature which is unobservable. (The total monopole is, of course, observable but its perturbation depends on the point identification with the background cosmology, i.e. is gauge-dependent.)

The various contributions to the temperature anisotropy have a simple interpretation in the Newtonian gauge. The term $\delta_\gamma/4$ can be thought of as intrinsic temperature variation (recall $\rho_\gamma \propto T^4$ for blackbody radiation at temperature T) over the background last-scattering surface[10]. The term ψ_E arises from the gravitational redshift when climbing out of a potential well at last scattering. The combination $\delta_\gamma/4 + \psi$ is often called the *Sachs-Wolfe* term after the authors of the pioneering study of redshift variations in a lumpy universe. The Doppler term $e_i v_b^i$ describes the blueshift from last scattering off electrons moving towards the observer. Finally, the *integrated Sachs-Wolfe* term describes the effect of gravitational redshifting from evolution of the potentials along the line of sight. Although we have performed the calculation in a specific gauge, the splitting of the temperature anisotropies into a Sachs-Wolfe, Doppler and integrated Sachs-Wolfe parts is gauge-invariant.

3.4.2 Qualitative features of the CMB power spectrum for adiabatic initial conditions

Large-scale behaviour

Consider modes outside the sound horizon at last scattering. From Eq. (3.3.80) we have

$$\dot{\delta}_\gamma + \frac{4}{3}\partial_i v_\gamma^i - 4\dot{\phi} = 0\,. \tag{3.4.23}$$

On large scales, the velocity $|\mathbf{v}_\gamma| \sim k\eta\phi$ (see Eq. 3.3.125) and so $\partial_i v_\gamma^i \sim k^2\eta\phi$ which is much smaller than any variation in ϕ through the matter-radiation transition where $\dot{\phi} \sim \mathcal{H}\phi \sim \phi/\eta$. The velocity divergence can therefore be ignored and we have

$$\delta_\gamma - 4\phi = \text{const.} \qquad \text{(large scales)}\,. \tag{3.4.24}$$

Recalling the adiabatic initial conditions, Eq. (3.3.123),

$$\delta_\gamma(0) = -2\phi(0)\,, \tag{3.4.25}$$

[10] A better interpretation, indicated by our derivation above, is that of an additional redshift due to the background expansion acting over the temporal perturbation to the true last-scattering surface.

the constant in Eq. (3.4.24) is $-6\phi(0) = 4\mathcal{R}(0)$, where $\mathcal{R}(0)$ is the initial value of the comoving curvature perturbation. At recombination, the universe is mostly matter dominated and so, on super-Hubble scales, $\psi(\eta_*) = \phi(\eta_*) = 9\phi(0)/10$ by the constancy of $\mathcal{R}$ (see Eq. 3.3.157). The Sachs-Wolfe term then becomes

$$\begin{aligned}\left(\frac{1}{4}\delta_\gamma + \psi\right)(\eta_*) &= \left(\frac{1}{4}\delta_\gamma - \phi\right)(\eta_*) + 2\phi(\eta_*) \\ &= \mathcal{R}(0) + 2\left(-\frac{3}{5}\right)\mathcal{R}(0) \\ &= -\frac{1}{5}\mathcal{R}(0) = \frac{1}{3}\phi(\eta_*)\,. \end{aligned} \tag{3.4.26}$$

The Doppler contribution to the temperature anisotropies is $e_i v_b^i = e_i v_\gamma^i \sim k\eta_*\phi$ and is sub-dominant on large scales by the ratio of the Hubble radius to the wavelength of the perturbation ($\sim k\eta_*$). The remaining contribution is from the integrated Sachs-Wolfe effect; this can be significant on large angular scales after the potentials start to decay and adds power incoherently to the Sachs-Wolfe contribution (see Fig. 6).

The contribution to the temperature anisotropy from the large-scale Sachs-Wolfe effect is therefore

$$\begin{aligned}\frac{\Delta T(\mathbf{e})}{\bar{T}_0} &= -\frac{1}{5}\mathcal{R}(0, -(\eta_0 - \eta_*)\mathbf{e}) \\ &= -\frac{1}{5}\int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\mathcal{R}(0,\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{e}(\eta_0-\eta_*)}\,. \end{aligned} \tag{3.4.27}$$

Using the Rayleigh plane-wave expansion,

$$e^{i\mathbf{k}\cdot\mathbf{x}} = 4\pi\sum_{lm} i^l j_l(kx) Y_{lm}^*(\hat{\mathbf{k}}) Y_{lm}(\hat{\mathbf{x}})\,, \tag{3.4.28}$$

and writing $-\mathbf{e} = \hat{\mathbf{n}}$, the observation direction, we have

$$\frac{\Delta T(\hat{\mathbf{n}})}{\bar{T}_0} = -\frac{4\pi}{5}\sum_{lm}\int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\mathcal{R}(0,\mathbf{k}) i^l j_l(k\chi_*) Y_{lm}^*(\hat{\mathbf{k}}) Y_{lm}(\hat{\mathbf{n}})\,, \tag{3.4.29}$$

where $\chi_* \equiv \eta_0 - \eta_*$ is the comoving distance back to last scattering. This is in the form of a spherical-harmonic expansion of the $\Delta T(\hat{\mathbf{n}})/\bar{T}_0$ with multipoles

$$a_{lm} = -\frac{4\pi}{5}\int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\mathcal{R}(0,\mathbf{k}) i^l j_l(k\chi_*) Y_{lm}^*(\hat{\mathbf{k}})\,. \tag{3.4.30}$$

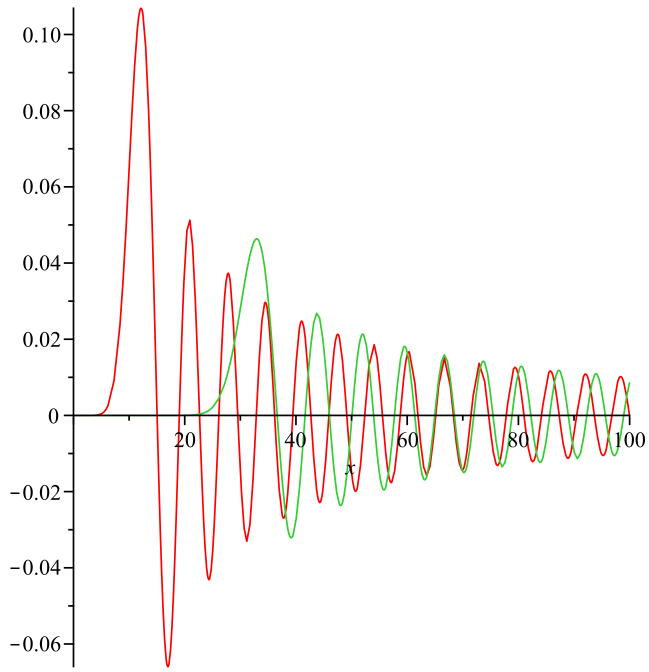

Figure 5: Spherical Bessel functions $j_l(x)$ for $l = 10$ (red) and $l = 30$ (green).

The two-point correlator for the multipoles is thus

$$
\begin{aligned}
\langle a_{lm} a^*_{l'm'} \rangle &= \left(\frac{4\pi}{5}\right)^2 \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{d^3\mathbf{k}'}{(2\pi)^{3/2}} \underbrace{\langle \mathcal{R}(0,\mathbf{k}) \mathcal{R}(0,\mathbf{k}')^* \rangle}_{\frac{2\pi^2}{k^3}\mathcal{P}_\mathcal{R}(k)\delta(\mathbf{k}-\mathbf{k}')} i^{l-l'} j_l(k\chi_*) j_{l'}(k'\chi_*) Y^*_{lm}(\hat{\mathbf{k}}) Y_{l'm'}(\hat{\mathbf{k}}') \\
&= \left(\frac{4\pi}{5}\right)^2 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{2\pi^2}{k^3} \mathcal{P}_\mathcal{R}(k) j_l(k\chi_*) j_{l'}(k\chi_*) Y^*_{lm}(\hat{\mathbf{k}}) Y_{l'm'}(\hat{\mathbf{k}}) \\
&= \delta_{ll'} \delta_{mm'} \underbrace{\frac{4\pi}{25} \int d\ln k \, \mathcal{P}_\mathcal{R}(k) j_l^2(k\chi_*)}_{C_l} . \qquad (3.4.31)
\end{aligned}
$$

As expected, the CMB anisotropies form a statistically-isotropic random process since they derive from projection of a statistically-homogeneous and isotropic field.

The spherical Bessel function appears in the expression for the power spectrum due to the projection of linear scales with wavenumber k onto angular scales (with corresponding "wavenumber" l). A mode with its wavevector perpendicular to the line of sight projects, at comoving distance χ_*, onto a modulation on the sky with angular wavelength $2\pi/(k\chi_*)$ corresponding to $l \sim k\chi_*$. Accordingly, the spherical Bessel functions peak sharply at argument $k\chi_* \approx l$ for large l (see Fig. 5). For smaller values of the argument, j_l has very low amplitude, while for larger arguments it oscillates. This

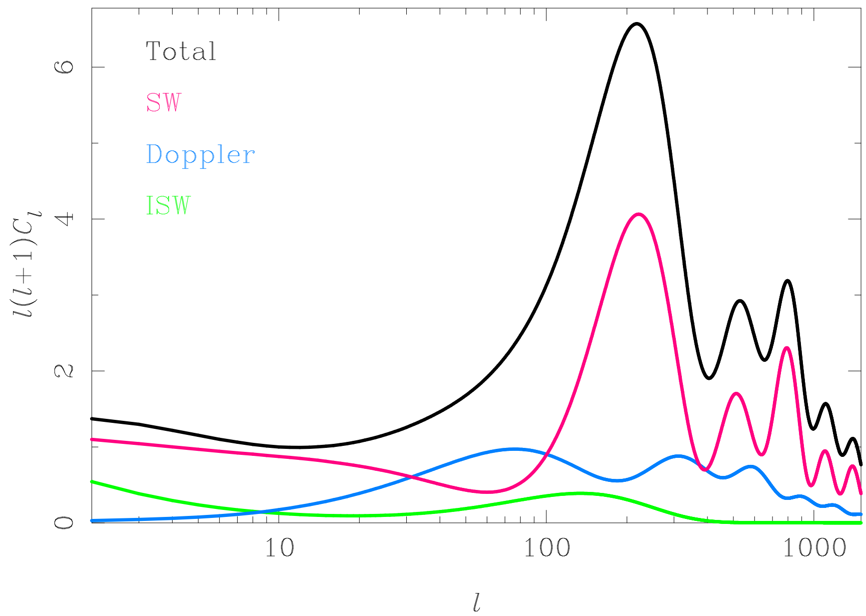

Figure 6: Contribution of the various terms in Eq. (3.4.22) to the temperature-anisotropy power spectrum from adiabatic initial conditions: $\delta_\gamma/4+\psi$ (denoted SW for Sachs-Wolfe; magenta); Doppler effect from v_b (blue); and the integrated Sachs-Wolfe effect (ISW; green).

behaviour arises since modes with wavenumbers that are not perpendicular to the line of sight project to angles larger than $2\pi/(k\chi_*)$.

If we make use of the standard integral

$$\int_0^\infty j_l^2(x)\,dx = \frac{1}{2l(l+1)} \tag{3.4.32}$$

for $l > 0$, we see that a scale-invariant primordial spectrum, for which $\mathcal{P}_\mathcal{R}(k) = A_s$ is a constant, gives a scale-invariant angular power spectrum

$$\frac{l(l+1)C_l}{2\pi} = \frac{1}{25}A_s\,. \tag{3.4.33}$$

More generally, a primordial spectrum that varies as a power-law in k (with some spectral index $n_s - 1$) gives an angular power spectrum going like

$$C_l \sim \frac{\Gamma(l+n_s/2-1/2)}{\Gamma(l-n_s/2+5/2)}\,, \tag{3.4.34}$$

where $\Gamma(x)$ is the Gamma function. We see that the CMB power spectrum on large scales is directly related to the amplitude and slope of the primordial power spectrum[11].

[11]Things are actually a little more complicated because of the integrated Sachs-Wolfe effect.

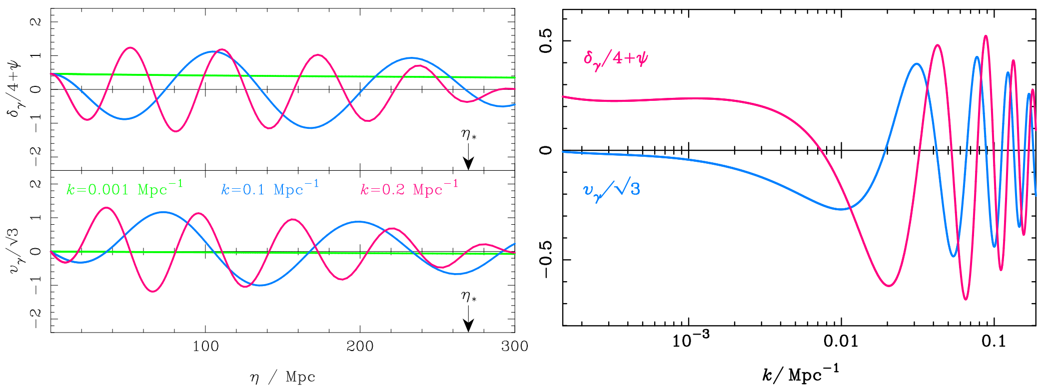

Figure 7: Evolution of the combination $\delta_\gamma/4+\psi$ (top left) and the photon velocity $|\mathbf{v}_\gamma|$ (bottom left) which determine the temperature anisotropies produced at last scattering (denoted by the arrow at η_*). Three modes are shown with wavenumbers $k = 0.001$, 0.1 and $0.2\,\mathrm{Mpc}^{-1}$, and the initial conditions are adiabatic. The fluctuations at the time of last scattering are shown as a function of wavenumber in the right-hand plot.

For an almost scale-free primordial spectrum ($n_s \approx 1$), we expect a plateau in C_l at small l (see Fig. 6). The latest measurements from WMAP indicate that $\mathcal{P}_\mathcal{R}(k)$ is close to scale-invariant, with $A_s = 2.4\times 10^{-9}$ at $k = 0.002\,\mathrm{Mpc}^{-1}$, but there is 2σ "evidence" for $n_s < 1$ (see Section 3.5).

Intermediate angular scales: acoustic peaks

On scales below the sound horizon at last scattering, the photons and baryons undergo acoustic oscillations. We showed in Section 3.3.5 that, for adiabatic initial conditions, the asymptotic form of the oscillation is a cosine mode, $\delta_\gamma \sim \cos(kr_s)$ where r_s is the sound horizon which is just $\eta/\sqrt{3}$ to the extent that baryon inertia can be ignored. Modes with $kr_s(\eta_*) = n\pi$ are at an extremum at last scattering (see Fig. 7) and, since there is lots of fluctuation power on such scales, they produce peaks in the angular power spectrum through the Sachs-Wolfe term at

$$l \sim n\pi\chi_*/r_s(\eta_*)\,. \tag{3.4.35}$$

The peaks in the Sachs-Wolfe contribution can be seen in Fig. 6.

Baryons do play an important role in the acoustic oscillations after matter-radiation equality since their inertia is then no longer negligible. Baryons reduce the sound speed, and hence the sound horizon, so changing the positions (and spacing) of the acoustic peaks in C_l. Moreover, since the baryons provide inertia to the plasma but negligible pressure support, the plasma has to be compressed further to balance gravitational infall. This produces an asymmetry in the oscillations boosting the power in the compressional peaks ($n = 1, 3, 5, \ldots$) over the peaks with even n. By measuring the relative peak heights, current CMB data suggest $\Omega_b h^2 = 0.0227$.

The photon continuity equation (3.4.23) implies that the peculiar velocity of the photons, hence baryons, oscillates $\pi/2$ out of phase with the energy density so the velocity peaks at last scattering for those modes for which δ_γ is at the midpoint of an oscillation (see Fig. 7). Such modes give zero Sachs-Wolfe contribution but the Doppler contribution partially fills in the zeroes in C_l (see Fig. 6). For baryon-free oscillations, $|\mathbf{v}_\gamma|/\sqrt{3} \sim \delta_\gamma/4$ and we should expect comparable power on intermediate scales from the Doppler and Sachs-Wolfe effects. However, by reducing the sound speed, baryons also reduce the amplitude of the photon peculiar velocity and diminish the Doppler contribution to the angular power spectrum.

Small scales: damping

On small scales, the CMB anisotropies are exponentially damped due to photon diffusion. Before recombination, fluctuations with wavelengths comparable or smaller than the mean-free path of photons to Compton scattering are damped out as photons can diffuse out of overdense regions into neighbouring underdensities. How far will a photon have diffused by last scattering? To answer this, consider the comoving mean-free path of the photons, l, which is related to the Thomson cross section and the number density of free electrons via

$$l = \frac{1}{a n_e \sigma_{\mathrm{T}}} . \tag{3.4.36}$$

In some interval of conformal time, $d\eta$, a photon undergoes $d\eta/l$ scatterings and random walks through a mean-squared distance $l^2 d\eta/l = l d\eta$. The total mean-squared distance that a photon will have moved by such a random walk by the time η_* is therefore

$$\int_0^{\eta_*} \frac{d\eta'}{a n_e \sigma_{\mathrm{T}}} \sim \frac{1}{k_D^2} , \tag{3.4.37}$$

which defines a damping scale k_D^{-1}. The photon density and velocity perturbations at wavenumber k are damped exponentially by photon diffusion, going like e^{-k^2/k_D^2}; this produces a similar damping in the angular power spectrum C_l (see Fig. 6). Evaluating k_D around recombination gives a comoving damping scale $\sim 30\,\mathrm{Mpc}$.

The predictions of linear perturbation theory for the angular power spectrum of CMB temperature anisotropies from adiabatic, scalar perturbations are in staggering agreement with observations (see Fig. 8).

3.5 Fluctuations from inflation

Inflation was originally proposed to solve shortcomings of the big bang model, but it was quickly realised that inflation should also naturally produce fluctuations in the

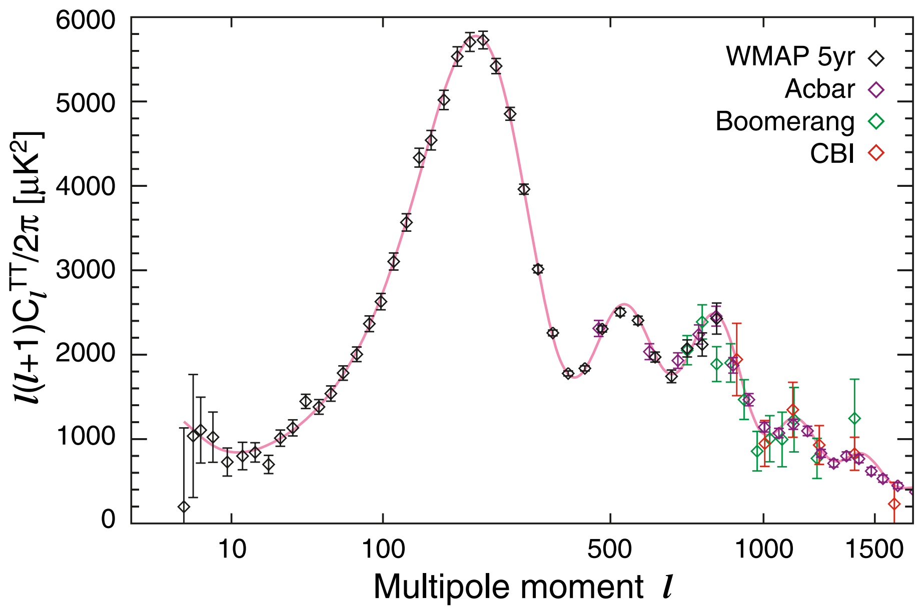

Figure 8: Recent measurements of the angular power spectrum of the temperature anisotropies. The solid line is the best-fit model in a ΛCDM universe with scalar perturbations evolving from adiabatic initial conditions. Credit: WMAP science team.

geometry of spacetime and the post-inflationary matter distribution. The key to seeing how this arises is to note that the comoving Hubble radius, $\mathcal{H}^{-1}$, *decreases* in time during inflation since

$$\frac{d\mathcal{H}^{-1}}{dt} = \frac{d}{dt}\left(\frac{da}{dt}\right)^{-1} = -\left(\frac{da}{dt}\right)^{-2}\frac{d^2a}{dt^2} < 0 \tag{3.5.1}$$

during accelerated expansion. This means that fluctuations of fixed comoving scale exit the Hubble radius during inflation. Equivalently, the physical Hubble radius, H^{-1}, is almost constant during slow-roll inflation but the physical wavelength of fluctuations is stretched quasi-exponentially and so pass outside the Hubble radius. Critically, this means that quantum fluctuations on (physically) microscopic scales are stretched up to cosmologically significant scales during inflation.

Consider a fluctuation with comoving wavenumber k. It exits the Hubble radius during inflation at some time t_k when $a = a_k$ and $H = H_k$ such that $k = a_kH_k$. During the post-inflationary deceleration of the universe, this mode will re-enter the Hubble radius. The ratio of the current Hubble radius, H_0^{-1}, to the current physical wavelength of the fluctuation is

$$\frac{k}{a_0H_0} = \frac{a_kH_k}{a_0H_0} = \frac{a_k}{a_e}\frac{a_e}{a_0}\frac{H_k}{H_0}, \tag{3.5.2}$$

where a_e is the scale factor at the end of inflation. The quantity $N(k) \equiv \ln(a_e/a_k)$ is the number of e-folds of inflation that occur after the mode of wavenumber k exits the

Hubble radius. It follows that

$$\begin{aligned}\frac{k}{a_0 H_0} &= e^{-N(k)}\frac{a_e}{a_0}\left(\frac{H_k^2}{H_0^2}\right)^{1/2} \\ &\approx e^{-N(k)}\frac{a_e}{a_0}\left(\frac{V}{\rho_{\mathrm{crit},0}}\right)^{1/2}, \end{aligned} \tag{3.5.3}$$

where we have used the Friedmann equation in a flat universe (appropriate once inflation is appreciably underway) dominated by the potential energy, V, of a scalar field – $3H_k^2 = V/M_{\mathrm{Pl}}^2$ – and introduced the critical density today via $3H_0^2 = 8\pi G\rho_{\mathrm{crit},0} = \rho_{\mathrm{crit},0}/M_{\mathrm{Pl}}^2$. If we assume that the energy density at the end of inflation is approximately V (i.e. that the Hubble rate changes little between Hubble exit and the end of inflation), and that all of this energy converts instantaneously (or at least over very few e-folds) into relativistic species (i.e. radiation), then $a_e/a_0 \approx (\rho_{\mathrm{rad},0}/V)^{1/4}$ where $\rho_{\mathrm{rad},0}$ is the current radiation density. We then have, from Eq. (3.5.3), that

$$\begin{aligned}N(k) &= -\ln\left(\frac{k}{a_0 H_0}\right) + \ln\left(\frac{\rho_{\mathrm{rad},0}}{V}\right)^{1/4} + \ln\left(\frac{V}{\rho_{\mathrm{crit},0}}\right)^{1/2} \\ &= -\ln\left(\frac{k}{a_0 H_0}\right) + \ln\left(\frac{V^{1/4}}{10^{16}\,\mathrm{GeV}}\right) + 55\,. \end{aligned} \tag{3.5.4}$$

So, for inflation near the GUT scale ($\sim 10^{16}\,\mathrm{GeV}$), fluctuations that are currently at the Hubble scale exited the Hubble radius 55 e-folds before the end of inflation.

3.5.1 Quantisation of a light scalar field in slow-roll inflation

During slow-roll inflation, the Hubble parameter is almost constant and the spatial sections are flat. The spacetime is therefore almost de Sitter. We shall begin by considering the quantum fluctuations of a light scalar field in a quasi-de Sitter Robertson-Walker universe ignoring the fluctuations in the spacetime geometry. This is clearly inconsistent with having fluctuations in the scalar field but we shall explain later how our result relates to a full calculation of the coupled fluctuations in the matter and the metric.

The action for a scalar field, denoted here by Φ to avoid confusion with the metric fluctuation ϕ, is, in conformal time and comoving coordinates,

$$S = \int d\eta d^3\mathbf{x}\left[\frac{1}{2}a^2\left(\dot{\Phi}^2 - (\boldsymbol{\nabla}\Phi)^2\right) - a^4 V(\Phi)\right]. \tag{3.5.5}$$

This follows from Eq. (1.9.9). In particular, the metric is $a^2\eta_{\mu\nu}$ so $\sqrt{-g} = a^4$ and $g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi = a^{-2}\eta^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi$.

In the background cosmology, $\Phi = \bar{\Phi}(\eta)$ is homogeneous. Its equation of motion follows from varying the action in Eq. (3.5.5):

$$\ddot{\bar{\Phi}} + 2\mathcal{H}\dot{\bar{\Phi}} + a^2 V' = 0\,, \tag{3.5.6}$$

where $V' \equiv dV/d\Phi$. Note that Eq. (3.5.6) is the conformal-time version of Eq. (1.9.11).

Consider now fluctuations in Φ. It is convenient to write $\Phi = \bar{\Phi} + u/a$ where $u = a\delta\Phi$. Expanding the action to second-order in u (which is what is required to get the field equations to linear-order in the fluctuation u) we get the sum of the zero-order action, a term $S^{(1)}$ that is first-order in u, and a term $S^{(2)}$ that is second-order in u. The first-order part is

$$\begin{aligned} S^{(1)} &= \int d\eta d^3\mathbf{x}\, \left[a\dot{\bar{\Phi}}\dot{u} - \dot{a}\dot{\bar{\Phi}}u - a^3 V'(\bar{\Phi})u\right] \\ &\sim -\int d\eta d^3\mathbf{x} \underbrace{[\partial_\eta(a\dot{\bar{\Phi}}) + \dot{\bar{\Phi}}\dot{a} + a^3 V'(\bar{\Phi})]}_{a\left[\ddot{\bar{\Phi}} + 2\mathcal{H}\dot{\bar{\Phi}} + a^2 V'(\bar{\Phi})\right]} u\,, \end{aligned} \tag{3.5.7}$$

where we have integrated by parts in the second line and dropped the boundary term since this does not contribute to the variation of the action. We see that the first-order action vanishes by virtue of the background field equation, as expected. This leaves the second-order action,

$$S^{(2)} = \frac{1}{2}\int d\eta d^3\mathbf{x}\, \left[\dot{u}^2 - 2\mathcal{H}u\dot{u} + \left(\mathcal{H}^2 - a^2 V''(\bar{\Phi})\right)u^2 - (\boldsymbol{\nabla}u)^2\right]\,. \tag{3.5.8}$$

Integrating the $\dot{u}$ term by parts, and dropping the boundary term, gives

$$\begin{aligned} S^{(2)} &= \frac{1}{2}\int d\eta d^3\mathbf{x}\, \left[\dot{u}^2 + \left(\dot{\mathcal{H}} + \mathcal{H}^2 - a^2 V''(\bar{\Phi})\right)u^2 - (\boldsymbol{\nabla}u)^2\right] \\ &= \frac{1}{2}\int d\eta d^3\mathbf{x}\, \left[\dot{u}^2 + \left(\ddot{a}/a - a^2 V''(\bar{\Phi})\right)u^2 - (\boldsymbol{\nabla}u)^2\right]\,. \end{aligned} \tag{3.5.9}$$

During slow-roll inflation, the effective squared mass of the inflaton, $m_{\text{eff}}^2 \equiv V''(\bar{\Phi})$ is much less than the Hubble rate squared. This follows since

$$\frac{m_{\text{eff}}^2}{H^2} = \frac{V''(\bar{\Phi})}{H^2} \approx \frac{3M_{\text{Pl}}^2 V''(\bar{\Phi})}{V(\bar{\Phi})} \equiv 3\eta_V\,, \tag{3.5.10}$$

where, recall from Eq. (1.9.20) that $3H^2 \approx V/M_{\text{Pl}}^2$, and the slow-roll parameter η_V was introduced in Eq. (1.9.18). Since $|\eta_V| \ll 1$ during slow-roll inflation, we see that $m_{\text{eff}}^2 \ll H^2$. Comparing the typical size of $\ddot{a}/a$ and $a^2 m_{\text{eff}}^2$ that multiply u^2 in the second-order action, we see that for quasi-de Sitter space,

$$\dot{a} = a^2 H \quad \Rightarrow \quad \frac{\ddot{a}}{a} \approx 2\dot{a}H = 2a^2H^2 \gg a^2 m_{\text{eff}}^2\,. \tag{3.5.11}$$

We can therefore drop the effective mass term from the action, working instead with

$$S^{(2)} \approx \int d\eta d^3\mathbf{x} \underbrace{\frac{1}{2}\left[\dot{u}^2 + (\ddot{a}/a)u^2 - (\boldsymbol{\nabla} u)^2\right]}_{\mathcal{L}} . \tag{3.5.12}$$

Apart from the (important!) term $(\ddot{a}/a)u^2$, this looks like the action for a free field in Minkowski space with quadratic Lagrangian $\mathcal{L}$. The equation of motion for u follows from varying the action in Eq. (3.5.12):

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial u} &= \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial(\partial_\mu u)} \\ \Rightarrow \quad \frac{\ddot{a}}{a}u &= \partial_\eta \dot{u} - \partial_i(\delta^{ij}\partial_j u) \\ \Rightarrow \quad 0 &= \ddot{u} - \frac{\ddot{a}}{a}u - \nabla^2 u\,. \end{aligned} \tag{3.5.13}$$

Canonical quantization

We now aim to quantize the field u following the standard methods of canonical quantization from quantum field theory. We first define the momentum conjugate to u by

$$\pi_u \equiv \frac{\partial \mathcal{L}}{\partial \dot{u}} = \dot{u}\,. \tag{3.5.14}$$

We then promote π and u to operator-valued, Hermitian fields, $\hat{\pi}(\eta,\mathbf{x})$ and $\hat{u}(\eta,\mathbf{x})$ that satisfy the *equal-time commutation relations*

$$[\hat{u}(\eta,\mathbf{x}), \hat{\pi}(\eta,\mathbf{x}')] = i\delta(\mathbf{x}-\mathbf{x}') \tag{3.5.15}$$

(setting $\hbar = 1$). We shall work in the *Heisenberg picture* where the quantum state of the system is constant in time but the operators evolve satisfying the quantum analogues of their classical equations of motion. For $\hat{u}(\eta,\mathbf{x})$, we have

$$\frac{\partial^2 \hat{u}}{\partial \eta^2} - \frac{\ddot{a}}{a}\hat{u} - \nabla^2\hat{u} = 0\,. \tag{3.5.16}$$

The general solution of this equation is of the form

$$\hat{u}(\eta,\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\left[\hat{a}(\mathbf{k})u_k(\eta)e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{a}^\dagger(\mathbf{k})u_k^*(\eta)e^{-i\mathbf{k}\cdot\mathbf{x}}\right] , \tag{3.5.17}$$

where $\hat{a}(\mathbf{k})$ is a time-independent operator, $\hat{a}^\dagger(\mathbf{k})$ is its Hermitian conjugate, and $u_k(\eta)$ is a complex-valued solution of the Fourier-space version of the classical equation of motion:

$$\ddot{u}_k(\eta) + (k^2 - \ddot{a}/a)u_k(\eta) = 0\,. \tag{3.5.18}$$

Note that $u_k(\eta)$ should be chosen so that u_k and u_k^* are linearly independent for Eq. (3.5.17) to be the general Hermitian solution. Note further that the mode functions only depend on $k = |\mathbf{k}|$. The momentum $\hat{\pi}(\eta, \mathbf{x}) = \partial_\eta \hat{u}(\eta, \mathbf{x})$ is then determined by differentiation.

The operators $\hat{a}(\mathbf{k})$ are constrained by the requirement that the fields $\hat{u}$ and $\hat{\pi}$ satisfy the equal-time commutation relations in Eq. (3.5.15). The Fourier transform of $\hat{u}(\eta, \mathbf{x})$ is, from Eq. (3.5.17),

$$\hat{u}(\eta, \mathbf{k}) = \hat{a}(\mathbf{k}) u_k(\eta) + \hat{a}^\dagger(-\mathbf{k}) u_k^*(\eta) \,, \tag{3.5.19}$$

and, since $\hat{\pi}(\eta, \mathbf{x}) = \partial_\eta \hat{u}(\eta, \mathbf{x})$, we have

$$\hat{\pi}(\eta, \mathbf{k}) = \hat{a}(\mathbf{k}) \dot{u}_k(\eta) + \hat{a}^\dagger(-\mathbf{k}) \dot{u}_k^*(\eta) \,. \tag{3.5.20}$$

The fields $\hat{u}(\eta, \mathbf{x})$ and $\hat{\pi}(\eta, \mathbf{x})$ are Hermitian so that, in Fourier space, $\hat{u}^\dagger(\eta, \mathbf{k}) = \hat{u}(\eta, -\mathbf{k})$ and $\hat{\pi}^\dagger(\eta, \mathbf{k}) = \hat{\pi}(\eta, -\mathbf{k})$. We can solve Eqs (3.5.19) and (3.5.20) for $\hat{a}(\mathbf{k})$ and $\hat{a}^\dagger(-\mathbf{k})$ to find

$$\hat{a}(\mathbf{k}) = w_k^{-1} \left[\dot{u}_k^*(\eta) \hat{u}(\eta, \mathbf{k}) - u_k^*(\eta) \hat{\pi}(\eta, \mathbf{k}) \right] \,, \tag{3.5.21}$$

where $w_k \equiv u_k(\eta) \dot{u}_k^*(\eta) - \dot{u}_k(\eta) u_k^*(\eta)$ is the Wronskian of the mode functions. The Wronskian is time-independent since Eq. (3.5.18) for the mode functions has no first-derivative term; it is also imaginary. To form commutation relations for the $\hat{a}(\mathbf{k})$, we require the non-zero equal-time commutator

$$\begin{aligned}
[\hat{u}(\eta, \mathbf{k}), \hat{\pi}(\eta, \mathbf{k}')] &= \int \frac{d^3\mathbf{x}}{(2\pi)^{3/2}} \frac{d^3\mathbf{x}'}{(2\pi)^{3/2}} \underbrace{[\hat{u}(\eta, \mathbf{x}), \hat{\pi}(\eta, \mathbf{x}')]}_{i\delta(\mathbf{x}-\mathbf{x}')} e^{-i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{k}'\cdot\mathbf{x}'} \\
&= i \int \frac{d^3\mathbf{x}}{(2\pi)^3} e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} \\
&= i\delta(\mathbf{k} + \mathbf{k}') \,.
\end{aligned} \tag{3.5.22}$$

We then have

$$\begin{aligned}
[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')] &= \frac{1}{w_k w_{k'}^*} \left\{ -\dot{u}_k^*(\eta) u_{k'}(\eta) [\hat{u}(\eta, \mathbf{k}), \hat{\pi}^\dagger(\eta, \mathbf{k}')] - u_k^*(\eta) \dot{u}_{k'}(\eta) [\hat{\pi}(\eta, \mathbf{k}), \hat{u}^\dagger(\eta, \mathbf{k}')] \right\} \\
&= \frac{1}{w_k w_{k'}} \left\{ \dot{u}_k^*(\eta) u_{k'}(\eta) [\hat{u}(\eta, \mathbf{k}), \hat{\pi}(\eta, -\mathbf{k}')] - u_k^*(\eta) \dot{u}_{k'}(\eta) [\hat{u}(\eta, -\mathbf{k}'), \hat{\pi}(\eta, \mathbf{k})] \right\} \\
&= \frac{i}{w_k w_{k'}} \left\{ \dot{u}_k^*(\eta) u_{k'}(\eta) - u_k^*(\eta) \dot{u}_{k'}(\eta) \right\} \delta(\mathbf{k} - \mathbf{k}') \\
&= i w_k^{-1} \delta(\mathbf{k} - \mathbf{k}') \,.
\end{aligned} \tag{3.5.23}$$

We can also show that

$$[\hat{a}(\mathbf{k}), \hat{a}(\mathbf{k}')] = 0 \,. \tag{3.5.24}$$

It is conventional to choose the mode functions so that the Wronskian $w_k = i$, in which case Eq. (3.5.23) reduces to the canonical form $[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')] = \delta(\mathbf{k}-\mathbf{k}')$. In Minkowski space, the $\hat{a}^\dagger(\mathbf{k})$ and $\hat{a}(\mathbf{k})$ are the usual particle creation and annihilation operators, respectively.

Exercise: Verify the commutation relation in Eq. (3.5.24).

Well inside the Hubble radius during inflation, $k^2 \gg \ddot{a}/a$ and the canonically-normalised solutions of Eq. (3.5.18) are

$$u_k(\eta) = \frac{e^{-ik\eta}}{\sqrt{2k}} \qquad (k \gg \mathcal{H})\,. \tag{3.5.25}$$

(This solution can always be multiplied by a k-dependent phase factor but this can then be absorbed into $\hat{a}(\mathbf{k})$ without altering the canonical commutation relations.) These mode functions are the same as in Minkowksi space and so we recover the usual quantum field theory of a free field in Minkowski space on scales well inside the Hubble radius. We shall assume that the fluctuations of the inflaton are in their ground state, $|0\rangle$. This state is annihilated by all the $\hat{a}(\mathbf{k})$:

$$\hat{a}(\mathbf{k})|0\rangle = 0\,. \tag{3.5.26}$$

In the Heisenberg picture, the field remains in this quantum state as inflation proceeds.

Power spectrum

We can now find the power spectrum of $u(\eta, \mathbf{x})$ and hence of the inflaton fluctuations. We get the power spectrum, $\mathcal{P}_u(k)$ by computing the two-point correlator of the the field u in Fourier space (see Eq. 3.1.12). The correlator is here defined as the quantum expectation value so, in the vacuum state,

$$\langle 0|\hat{u}(\eta,\mathbf{k})\hat{u}^\dagger(\eta,\mathbf{k}')|0\rangle = \frac{2\pi^2}{k^3}\mathcal{P}_u(k)\delta(\mathbf{k}-\mathbf{k}')\,. \tag{3.5.27}$$

Noting that

$$\hat{u}^\dagger(\eta,\mathbf{k})|0\rangle = u_k^*(\eta)\hat{a}^\dagger(\mathbf{k})|0\rangle\,, \tag{3.5.28}$$

we have

$$\begin{aligned}\langle 0|\hat{u}(\eta,\mathbf{k})\hat{u}^\dagger(\eta,\mathbf{k}')|0\rangle &= u_k(\eta)u_{k'}^*(\eta)\langle 0|\hat{a}(\mathbf{k})\hat{a}^\dagger(\mathbf{k}')|0\rangle \\ &= u_k(\eta)u_{k'}^*(\eta)\langle 0|[\hat{a}(\mathbf{k}),\hat{a}^\dagger(\mathbf{k}')]|0\rangle \\ &= |u_k(\eta)|^2\delta(\mathbf{k}-\mathbf{k}')\,. \end{aligned} \tag{3.5.29}$$

The power spectrum of u is thus

$$\mathcal{P}_u(k) = \frac{k^3}{2\pi^2}|u_k(\eta)|^2 \,. \tag{3.5.30}$$

Finally, since u is related to the inflaton fluctuation by $u = a\delta\Phi$, the power spectrum of $\delta\Phi$ is

$$\mathcal{P}_{\delta\Phi}(k) = \frac{k^3}{2\pi^2}\left|\frac{u_k(\eta)}{a(\eta)}\right|^2 \,. \tag{3.5.31}$$

The equations of motion for the mode functions, Eq. (3.5.18), has a growing-mode solution $u \propto a$ well after Hubble exit ($k \ll \mathcal{H}$) and so the power spectrum of $\delta\Phi$ becomes constant outside the Hubble radius.

We require a more detailed solution for the mode functions to set the amplitude of the power spectrum. Well inside the Hubble radius, we have seen $u_k = e^{-ik\eta}/\sqrt{2k}$, while well outside $u \propto a$. We only need the evolution through Hubble exit to interpolate between these two limits. During slow-roll inflation the fractional variation in the Hubble rate per Hubble time is small so we shall treat H as a constant, H_k, for the few e-folds either side of Hubble exit. Then, in this interval

$$\dot{a}/a = \mathcal{H} = aH_k \quad \Rightarrow \quad a = -\frac{1}{H_k\eta} \,. \tag{3.5.32}$$

Here, $\eta \to -\infty$ at the start of inflation, $k\eta = -1$ at Hubble exit (since $k = \mathcal{H} = aH$ then) and $\eta \to 0_-$ in the infinite future. It follows that

$$\ddot{a} = -\frac{2}{H_k\eta^3} \quad \Rightarrow \quad \frac{\ddot{a}}{a} = \frac{2}{\eta^2} \,. \tag{3.5.33}$$

The mode equation is then

$$\ddot{u}_k + (k^2 - 2/\eta^2)u_k = 0 \,. \tag{3.5.34}$$

The exact solution with the required behaviour well inside the Hubble radius ($k\eta \to -\infty$) is

$$u_k(\eta) = \frac{e^{-ik\eta}}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right) \,. \tag{3.5.35}$$

A few e-folds after Hubble exit

$$u_k(\eta) \approx \frac{-ie^{-ik\eta}}{\eta\sqrt{2k^3}} \quad \text{so} \quad \frac{u_k(\eta)}{a(\eta)} \approx \frac{iH_k e^{-ik\eta}}{\sqrt{2k^3}} \,. \tag{3.5.36}$$

It follows that, for modes that have exited the Hubble radius when the Hubble rate is H_k, the power spectrum of the inflaton fluctuations is constant in time and equal to

$$\mathcal{P}_{\delta\Phi}(k) = \left(\frac{H_k}{2\pi}\right)^2 \,. \tag{3.5.37}$$

Since H varies only slowly during inflation, H_k and hence $\mathcal{P}_{\delta\Phi}(k)$ depend only weakly on k. This simple result is very important; it states that *a light scalar field in quasi-de Sitter spacetime acquires an almost-scale-invariant spectrum of fluctuations with amplitude* $(H_k/2\pi)^2$. The spectrum would be exactly scale-invariant were it not for the small decrease in the Hubble rate during inflation.

3.5.2 The comoving curvature perturbation from slow-roll inflation

At the end of inflation, the inflaton decays into other particles reheating the universe. Treating the decay as instantaneous on a hypersurface of uniform Φ – we shall see shortly that such hypersurfaces are comoving and, by the Poisson equation (3.3.106), have uniform energy density – all the decay products of the inflaton will also be uniformly distributed across this surface. This means that the primordial fluctuations in the reheated universe will be *adiabatic* and are fully characterised by their (conserved) comoving curvature perturbation, $\mathcal{R}$.

So, our aim is to calculate $\mathcal{R}$ but the calculation of the previous section ignored the very metric fluctuations on which $\mathcal{R}$ depends. A more careful calculation, computing the action to second-order in perturbations for the coupled system of scalar metric and inflaton field fluctuations shows that our previous analysis is still valid provided that we interpret $\delta\Phi$ as the field fluctuations on hypersurfaces with zero intrinsic curvature[12]. We can relate $\delta\Phi$ on zero-curvature hypersurfaces to the comoving curvature perturbation as follows. The gauge-invariant expression for $\mathcal{R}$ is, from Eq. (3.3.149),

$$\mathcal{R} = -\phi - \frac{1}{3}\nabla^2 E + \mathcal{H}(B+v)\,. \tag{3.5.38}$$

We can eliminate the peculiar velocity v in terms of the inflaton field fluctuation by noting that, in any gauge,

$$T^i{}_0 = q^i = (\bar\rho + \bar P)\delta^{ij}\partial_j v\,. \tag{3.5.39}$$

The stress-energy tensor for a scalar field was given in Eq. (1.9.2), which we write here as

$$T^\mu{}_\nu = \nabla^\mu\Phi\nabla_\nu\Phi - \delta^\mu{}_\nu\left(\frac{1}{2}(\nabla_\rho\Phi)^2 - V\right)\,. \tag{3.5.40}$$

[12]The action for the coupled system is equivalent to Eq. (3.5.9) up to terms that vanish with the slow-roll parameters. In detail, the term $\ddot a/a - V''(\bar\Phi)$ gets replaced with $\ddot z/z$ where $z \equiv a\dot{\bar\Phi}/\mathcal{H}$. To leading-order in slow-roll, both terms reduce to $\ddot a/a$.

It follows that

$$\begin{aligned} q^i &= g^{i\mu}\partial_\mu \Phi \partial_\eta \Phi \\ &= g^{i0}\dot{\Phi}^2 + g^{ij}\dot{\bar{\Phi}}\partial_j \Phi \\ &= -a^{-2}B^i\dot{\bar{\Phi}}^2 - a^{-2}\dot{\bar{\Phi}}\delta^{ij}\partial_j \delta\Phi \\ &= -a^{-2}\dot{\bar{\Phi}}^2\delta^{ij}\partial_j \left(B + \delta\Phi/\dot{\bar{\Phi}}\right) . \end{aligned} \tag{3.5.41}$$

(Here, we have used $\delta g^{\mu\nu} = -\bar{g}^{\mu\alpha}\delta g_{\alpha\beta}\bar{g}^{\beta\nu}$ which follows from the general result for the perturbation of the inverse of a matrix.) Since $\bar{\rho}+\bar{P} = a^{-2}\dot{\bar{\Phi}}^2$ (see Eqs 1.9.3 and 1.9.4), we have

$$a^{-2}\dot{\bar{\Phi}}^2\delta^{ij}\partial_j v = -a^{-2}\dot{\bar{\Phi}}^2\delta^{ij}\partial_j \left(B + \delta\Phi/\dot{\bar{\Phi}}\right) \quad \Rightarrow \quad B + v = -\delta\Phi/\dot{\bar{\Phi}} . \tag{3.5.42}$$

We can thus write the comoving curvature perturbation for a universe dominated by a scalar field in gauge-invariant form as

$$\mathcal{R} = -\phi - \frac{1}{3}\nabla^2 E - \frac{\mathcal{H}\delta\Phi}{\dot{\bar{\Phi}}} . \tag{3.5.43}$$

It is straightforward to verify that this is gauge invariant using Eqs (3.3.59) and (3.3.61) and noting that Φ is a Lorentz scalar so its perturbation transforms as $\delta\Phi \to \delta\Phi - T\dot{\bar{\Phi}}$. Note that the comoving gauge has $q^i = B^i = 0$ which imply $\delta\Phi = 0$, as previously advertised.

We have computed the quantum fluctuations of Φ in the zero-curvature gauge, in which the hypersurfaces of constant time have zero intrinsic curvature. From Eq. (3.3.148), $\phi + \nabla^2 E/3 = 0$ in this gauge and so

$$\mathcal{R} = -\mathcal{H}\delta\Phi/\dot{\bar{\Phi}} \qquad \text{(zero-curvature gauge)} . \tag{3.5.44}$$

It follows from Eq. (3.5.37) that the power spectrum of $\mathcal{R}$ after horizon crossing is

$$\mathcal{P}_{\mathcal{R}}(k) = \left(\frac{\mathcal{H}}{\dot{\bar{\Phi}}}\right)^2 \mathcal{P}_{\delta\Phi}(k) = \left(\frac{H^2}{2\pi\partial_t\bar{\Phi}}\right)^2 . \tag{3.5.45}$$

The right-hand side should be evaluated at Hubble exit for the given k[13].

[13]Our earlier calculation showed that $\delta\Phi$ was constant outside the horizon. This is not quite correct when we include the effect of metric perturbations and the effective mass of the field. The correct result is that $\delta\Phi \propto \dot{\bar{\Phi}}/\mathcal{H}$ outside the Hubble radius, so that $\mathcal{R}$ is constant. This is why it is correct to evaluate the $H/\partial_t\bar{\Phi}$ term in Eq. (3.5.45) at Hubble exit.

Slow-roll expansion

For a slowly-rolling scalar field, we can express $\mathcal{P}_{\mathcal{R}}(k)$ directly in terms of the inflaton potential. During slow roll, the potential energy of the field dominates over the kinetic energy and so

$$H^2 \approx \frac{1}{3M_{\mathrm{Pl}^2}} V(\bar{\Phi}) , \tag{3.5.46}$$

and the field evolution is friction limited:

$$3H\partial_t \bar{\Phi} = -V'(\bar{\Phi}) . \tag{3.5.47}$$

(See Eqs 1.9.20 and 1.9.21). It follows that

$$\begin{aligned} \mathcal{P}_{\mathcal{R}}(k) = \left(\frac{H^2}{2\pi\partial_t\bar{\Phi}}\right)^2 \approx \left(\frac{3H^3}{2\pi V'}\right)^2 &\approx \frac{(V/M_{\mathrm{Pl}}^2)^3}{3(2\pi)^2(V')^2} \\ &= \frac{8}{3}\left(\frac{V^{1/4}}{\sqrt{8\pi}M_{\mathrm{Pl}}}\right)^4 \frac{1}{\epsilon_V} , \end{aligned} \tag{3.5.48}$$

where the slow-roll parameter

$$\epsilon_V \equiv \frac{M_{\mathrm{Pl}}^2}{2}\left(\frac{V'}{V}\right)^2 \tag{3.5.49}$$

was introduced in Eq. (1.9.17). The large-angle CMB observations constrain $\mathcal{P}_{\mathcal{R}}(k) \sim 2 \times 10^{-9}$ on current Hubble scales [via Eq. (3.4.33)]. It follows that

$$V^{1/4} \sim 6 \times 10^{16} \epsilon_V^{1/4} \, \mathrm{GeV} . \tag{3.5.50}$$

The quantity $V^{1/4}$ describes the *energy scale of inflation* and, since $\epsilon_V \ll 1$, the energy scale is at least two orders of magnitude below the Planck scale ($\sim 10^{19}$ GeV). It is, however, plausible, that inflation occurred around the GUT scale, $\sim 10^{16}$ GeV.

Spectral index of the primordial power spectrum

We have already noted that slow-roll inflation produces a spectrum of curvature perturbations that is almost scale-invariant. We can quantify the small departures from scale-invariance by forming the *spectral index* $n_s(k)$. Generally, this is a scale-dependent quantity defined by

$$n_s(k) - 1 \equiv \frac{d \ln \mathcal{P}_{\mathcal{R}}(k)}{d \ln k} , \tag{3.5.51}$$

where the -1 is conventional and (unfortunately!) means that a scale-free spectrum has $n_s = 1$. For a constant n_s, this definition implies a power-law spectrum

$$\mathcal{P}_{\mathcal{R}}(k) = A_s (k/k_{\mathrm{pivot}})^{n_s - 1} \tag{3.5.52}$$

for some pivot scale $k_{\rm pivot}$.

We can evaluate n_s by noting that

$$\frac{d}{d\ln k} = \frac{dt}{d\ln k}\frac{d\bar{\Phi}}{dt}\frac{d}{d\bar{\Phi}}\,, \tag{3.5.53}$$

and, since $k = aH = \partial_t a$ at Hubble exit,

$$\frac{d\ln k}{dt} = \frac{\partial_t^2 a}{k} = \frac{1}{H}\frac{\partial_t^2 a}{a} \approx H(1-\epsilon_V) \tag{3.5.54}$$

from Eq. (1.9.23). We thus have, to leading-order in the slow-roll parameters,

$$\begin{aligned}\frac{d}{d\ln k} &\approx \frac{1}{H}\frac{d\bar{\Phi}}{dt}\frac{d}{d\bar{\Phi}} \\ &\approx -\frac{V'}{3H^2}\frac{d}{d\bar{\Phi}} \\ &\approx -M_{\rm Pl}^2\frac{V'}{V}\frac{d}{d\bar{\Phi}} \\ &\approx -M_{\rm Pl}\sqrt{2\epsilon_V}\frac{d}{d\bar{\Phi}}\,.\end{aligned} \tag{3.5.55}$$

We can now differentiate Eq. (3.5.48) to find

$$\begin{aligned}n_s - 1 &= -M_{\rm Pl}\sqrt{2\epsilon_V}\frac{d}{d\bar{\Phi}}\left(\ln V - \ln\epsilon_V\right) \\ &= -M_{\rm Pl}\sqrt{2\epsilon_V}\left(\frac{V'}{V} - \frac{\epsilon_V'}{\epsilon_V}\right)\,.\end{aligned} \tag{3.5.56}$$

The derivative of ϵ_V is

$$\begin{aligned}\frac{d\ln\epsilon_V}{d\bar{\Phi}} &= 2\left(\frac{V''}{V'} - \frac{V'}{V}\right) \\ &\approx \frac{\sqrt{2}}{M_{\rm Pl}}\left(\frac{\eta_V}{\sqrt{\epsilon_V}} - 2\sqrt{\epsilon_V}\right)\,,\end{aligned} \tag{3.5.57}$$

where η_V is the slow-roll parameter related to the curvature of the potential; see Eq. (3.5.10). This gives the final, simple result

$$n_s(k) - 1 = 2\eta_V(\bar{\Phi}) - 6\epsilon_V(\bar{\Phi})\,. \tag{3.5.58}$$

We see that departures from scale-invariance are first order in the slow-roll parameters. It can be shown that $dn_s/d\ln k$ is second-order in slow roll so a power-law primordial power spectrum is a very good approximation for slow-roll inflation.

3.5.3 Gravitational waves from inflation

Gravitational waves (tensor modes) are also excited during inflation. Recall, from Section 3.3.1, that these are described by a metric of the form

$$ds^2 = a^2(\eta)\left[d\eta^2 - (\delta_{ij} + 2E^T_{ij})dx^i dx^j\right] , \tag{3.5.59}$$

where E^T_{ij} is trace-free and $\delta^{ik}\partial_k E^T_{ij} = 0$. There are two degrees of freedom associated with E^T_{ij} and these behave like massless scalar fields during inflation. Their quantum fluctuations are stretched up to cosmological scales during inflation and they acquire a power spectrum

$$\mathcal{P}_h(k) = \frac{8}{M^2_{\rm Pl}}\left(\frac{H_k}{2\pi}\right)^2 , \tag{3.5.60}$$

outside the Hubble radius, where the power spectrum conventions are such that

$$\langle (2E^T_{ij})(2E^{ij\,T})\rangle = \int d\ln k\, \mathcal{P}_h(k) . \tag{3.5.61}$$

Gravitational waves are thus a direct probe of the Hubble rate during inflation, or, using the slow-roll approximation,

$$\mathcal{P}_h(k) \approx \frac{128}{3}\left(\frac{V^{1/4}}{\sqrt{8\pi}M_{\rm Pl}}\right)^4 , \tag{3.5.62}$$

of the energy scale. Note that

$$r \equiv \frac{\mathcal{P}_h(k)}{\mathcal{P}_{\mathcal{R}}(k)} \approx 16\epsilon_V , \tag{3.5.63}$$

which defines the dimensionless tensor-to-scalar ratio r.

The spectrum of gravitational waves is almost scale-invariant with a spectral index

$$\begin{aligned} n_t \equiv \frac{d\ln \mathcal{P}_h(k)}{d\ln k} \approx \frac{d\ln V}{d\ln k} &= -M_{\rm Pl}\sqrt{2\epsilon_V}\frac{V'}{V} \\ &= -M_{\rm Pl}\sqrt{2\epsilon_V}\frac{\sqrt{2\epsilon_V}}{M_{\rm Pl}} \\ &= -2\epsilon_V . \end{aligned} \tag{3.5.64}$$

Note that this is always negative (the spectrum is said to be *red*) which is a direct consequence of the Hubble parameter falling as inflation proceeds. It follows that $r \approx -8n_t$ in slow-roll inflation which is an example of a slow-roll *consistency relation* between the spectra of curvature perturbations and gravitational waves.

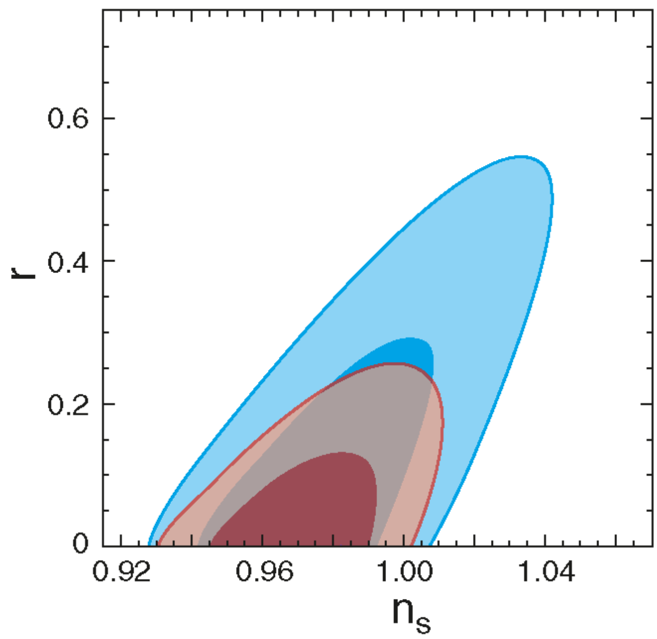

Figure 9: Constraints in the r-n_s plane from five years of WMAP data (blue) and WMAP plus distance information from supernovae and baryon acoustic oscillations (red). In each case the 68% and 95% contours are plotted. Note the degeneracy between r and n_s. Credit: WMAP science team.

3.5.4 Current observational constraints on r and n_s

In slow-roll inflation, the quantities r and n_s constrain the fractional gradient and curvature of the inflaton potential and so constrain its *shape*. The effect of increasing n_s is to enhance the power in cosmological observables, such as the CMB anisotropy power spectrum, on small scales over that on large scales. The CMB can also be used to constrain primordial gravitational waves, though future direct detection of gravitational waves from space may ultimately prove to be a more sensitive probe. Gravitational waves produce an anisotropic expansion of space (a shear) once they enter the Hubble radius which causes the redshift of a CMB photon to depend on its direction of propagation. This gives rise to temperature anisotropies but they are confined to large angular scales since the amplitude of gravitational waves falls away asymptotically as $1/a$ inside the Hubble radius. Since there are few independent modes in the CMB to measure on large angular scales, the accuracy of any determination of r via this route is limited[14].

[14]Polarization of the CMB is a more promising method for detecting primordial gravitational waves; see the Lent-term course *Advanced Cosmology*.

Current constraints in the r-n_s plane are plotted in Fig. 9. These parameters are rather degenerate in current data: increasing n_s and r boosts power on all scales and so, if the amplitude A_s is reduced, produces little net effect on the CMB angular power spectrum. The constraints on the individual parameters are $r < 0.22$ (95% confidence) and $n_s = 0.960 \pm 0.013$ (with 68% error interval). The result for n_s has been interpreted as supportive evidence that cosmological inflation really occurred. Other observations supporting inflation include:

- The curvature of the universe is measured to be close to zero: $-0.0179 < \Omega_K < 0.0081$;
- There is no strong evidence for non-Gaussianity of the primordial fluctuations[15];
- There are measured correlations between the CMB temperature anisotropies and polarization on large angular scales that can only have arisen from apparently acausal (i.e. super-Hubble) fluctuations around the time of recombination;
- There is no evidence that the primordial fluctuations were not adiabatic.

For many, detecting a background of primordial gravitational waves consistent with a red power spectrum would be conclusive evidence of inflation. The ultimate test of single-field, slow-roll inflation would be verification of the slow-roll consistency relation, $r = -8n_t$, but this is extremely challenging observationally.

[15]We have not had time to discuss the Gaussianity or otherwise of the primordial fluctuations here. Slow-roll inflation implies that the fluctuations are essentially free fields and, for the vacuum state $|0\rangle$, must therefore be Gaussian distributed. (The ground state wavefunction of the harmonic oscillator is a Gaussian function for the same reason.)