Chapter 27. Examples for Part VI.

CONTENT OF CHAPTER 27

27.1. Example 1. Rough estimates.

Assume that for an order of magnitude estimation the linear approximation gives more or less correct answer up to $\delta \simeq 1$. Estimate the ratio of $\delta(z)$ at $z = 99$ and $\delta(z)$ at $z = 9$, if it is given that growing and decaying modes are approximately equal to each other at $z = 999$.

Solution

According to the solution given in Section 26.3.

$$\frac{\delta(99)}{\delta(9)} \simeq \frac{A_+ 10^{-2} + A_- 10^3}{A_+ 10^{-1} + A_- 10^{3/2}} = \frac{10^{-2} + 10^3 f}{10^{-1} + 10^{3/2} f}, \tag{1}$$

where

$$f = \frac{A_-}{A_+}. \tag{2}$$

Taking into account that

$$A_+ 10^{-3} \simeq A_- 10^{9/2}, \tag{3}$$

we find that

$$f \simeq 10^{-3-9/2} = 10^{-15/2}, \tag{4}$$

hence

$$\frac{\delta(99)}{\delta(9)} \simeq \frac{10^{-2} + 10^{3-15/2}}{10^{-1} + 10^{3/2-15/2}} = \frac{10^{-2} + 10^{-9/2}}{10^{-1} + 10^{-6}} \approx 10^{-2+1} = 0.1. \tag{5}$$

27.2. Example 2. Amplitude of δ as a function of time. Alternative derivation.

Derive the equation for the evolution of small density perturbations, $\delta = (\rho' - \rho)/\rho$ after decoupling to show that

$$\ddot{\delta} + (4/3t)\dot{\delta} - (2/3t^2)\delta = 0.$$

Solution

Starting from

$$\ddot{R} = -\frac{4\pi G \rho R}{3},$$

perturb R and ρ: $R' = R(1 + h)$, and $\rho' = \rho(1 + \delta)$.

Putting this in the perturbed equation

$$\ddot{R}' = -\frac{4\pi G\rho' R'}{3},$$

we obtain

$$\ddot{R}(1+h) + 2\dot{R}\dot{h} + R\ddot{h} = -\frac{4\pi G\rho(1+\delta)R(1+h}{3}.$$

Using unperturbed equation, we obtain linearized equation

$$\ddot{R}h + 2\dot{R}\dot{h} + R\ddot{h} = -\frac{4\pi G\rho R(\delta+h)}{3}.$$

To relate h and δ we use the conservation of energy equation $\rho R^3 = \rho R^3(1+3h)(1+\delta)$, or $1 = 1 + 3h + \delta$, so $h = -\delta/3$.
Thus we have

$$-\ddot{R}\frac{\delta}{3} - \frac{2}{3}\dot{R} - R\frac{\ddot{\delta}}{3} = -\frac{4\pi G\rho}{3}R\frac{2}{3}\delta,$$

then

$$\ddot{\delta} + 2\frac{\dot{R}}{R} + \frac{\ddot{R}}{R}\delta = \frac{8\pi G\rho}{3}\delta.$$

For the dust-like Universe $R \sim t^{2/3}$, so

$$\frac{\dot{R}}{R} = \frac{2}{3t}, \quad \frac{\ddot{R}}{R} = \frac{2}{3}(\frac{2}{3}-1)t^{-2} = -\frac{2}{9t^2}.$$

From the unperturbed equation

$$\frac{8\pi G\rho}{3} = -\frac{2\ddot{R}}{R} = \frac{4}{9t^2}.$$

Then

$$\ddot{\delta} + 2\frac{2}{3t} + (-\frac{2}{9} - \frac{4}{9})\frac{\delta}{t^2} = 0,$$

and finally

$$\ddot{\delta} + \frac{4}{3t} - \frac{2}{3t^2}\delta = 0.$$

Taking trial solution $\delta = At^m$, we obtain

$$m(m-1) + \frac{4m}{3} - \frac{2}{3} = 0, \;\; 3m^2 + m - 2 = 0.$$

Solutions of this quadratic equation are

$$m = \frac{-1 \pm \sqrt{1+24}}{6} = \frac{-1 \pm 5}{6},$$

thus $m_+ = 2/3$ and $m_- = -1$ (growing and decaying modes). So we have

$$\delta = A_+(t/t_0)^{2/3} + A_-(t/t_0)^{-1}.$$