Lecture 9. *Last updated 07.03.10*

IX. BLACK HOLES

A. Limit of stationarity

Let us consider ds for the test particle in rest, i.e. put $dr = d\theta = d\phi = 0$, in this case

$$ds^2 = g_{00} dx^{0^2}, \tag{IX.1}$$

If $g_{00} = 0$ then $ds^2 = 0$, which means that the world line of the particle at rest is the world line of light, hence at the surface $g_{00} = 0$ no particle with finite rest mass can be at rest. Thus the surface $g_{00} = 0$ is called the limit of stationarity.

B. Event horizon

Let us consider a surface

$$F(r) = const \tag{IX.2}$$

and let

$$n_i = F_{,i} \tag{IX.3}$$

is its normal. If $g^{11} = 0$ then

$$g^{ik} n_i n_k = g^{11} n_1 n_1 = 0, \tag{IX.4}$$

which means that n_i is the null vector and any particle with finite rest mass can not move outward the surface $g^{11} = 0$, thus this surface is the event horizon.

C. Schwarzschild black holes

Schwarzschild Black holes are described by the following metric

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{r_g}{r}\right)} - r^2 \left(\sin^2 \theta d\phi^2 + d\theta^2\right), \tag{IX.5}$$

obtained in the previous lecture.One can see that both the limit of stationarity and the event horizon are located at $r = r_g$.
Let us consider the structure of light cone in the Schwarzschild metric using the new coordinates $c\tau$ and R introduced in Lecture 8. Putting $ds = 0$, we have

$$c\frac{d\tau}{dR} = \pm \frac{1}{\left(\frac{3}{2r_g}(R - c\tau)\right)^{1/3}} = \pm \sqrt{\frac{r_g}{r}}. \tag{IX.6}$$

Thus we can see that if $r > r_g$

$$|c\frac{d\tau}{dR}| < 1 \tag{IX.7}$$

and the surface $r = const$ is inside the light cone, while for $r < r_g$

$$|c\frac{d\tau}{dR}| > 1 \tag{IX.8}$$

and the surface $r = const$ is outside the light cone, which means that all particles and even photons should propagate inward. In order words we can see that the surface $r = r_g$ is the event horizon.

D. Kerr Black Holes

The Kerr metric describing the gravitational field of rotating black holes has the following form

$$ds^2 = (1 - \frac{r_g r}{\rho^2})c^2 dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - (r^2 + a^2 + \frac{r_g r a^2}{\rho^2} \sin^2 \theta) \sin^2 \theta d\phi^2 +$$

$$+ \frac{2 r_g r a}{\rho^2} \sin^2 \theta c d\phi dt, \tag{IX.9}$$

where

$$\rho^2 = r^2 + a^2 \cos^2 \theta, \ \ \Delta = r^2 - r_g r + a^2 \ \text{ and } \ a = \frac{J}{mc} \tag{IX.10}$$

and J is the specific angular momentum of the black hole.

1. *Limit of stationarity*

The location of the limit of stationarity, r_{st}, corresponding to $g_{00} = 0$, in the Kerr metric is determined from the equation

$$1 - \frac{r_g r}{\rho^2} = 0, \ \text{ thus } \ r^2 - r_g r + a^2 \cos^2 \theta = 0. \tag{IX.11}$$

Solving this equation we obtain that

$$r_{st} = \frac{1}{2}(r_g \pm \sqrt{r_g^2 - 4a^2 \cos^2 \theta}) = \frac{r_g}{2} \pm \sqrt{(\frac{r_g}{2})^2 - a^2 \cos^2 \theta}. \tag{IX.12}$$

2. *Event horizon*

The location of the event horizon, r_{hor} is determined by $g^{11} = 0$. In the Kerr metric this corresponds to $g_{11} = \infty$, i.e. corresponds to

$$\Delta = r^2 - r_g r + a^2 = 0, \tag{IX.13}$$

and

$$r = \frac{1}{2}(r_g \pm \sqrt{r_g^2 - 4a^2 \cos^2\theta}) = \frac{r_g}{2} \pm \sqrt{(\frac{r_g}{2})^2 - a^2 \cos^2\theta}, \tag{IX.14}$$

hence

$$r_{hor} = \frac{r_g}{2} \pm \sqrt{\left(\frac{r_g}{2}\right)^2 - a^2}. \tag{IX.15}$$

E. "Ergosphere" and Penrose process

1. *Ergosphere*

The region between the limit of stationarity and the event horizon is called the "ergosphere". By the Penrose process it is possible to extract the rotational energy of the Kerr black hole.

2. *Penrose process*

The Penrose process is a process wherein energy can be extracted from a rotating black hole. That extraction is made possible because the rotational energy of the black hole is located not inside the event horizon, but outside in a curl gravitational field. Such field is also called gravi-magnetic field. All objects in the ergosphere are unavoidably dragged by the rotating spacetime. Imagine that some body enters into the black hole and then it is split there into two pieces. The momentum of the two pieces of matter can be arranged so that one piece escapes to infinity, whilst the other falls past the outer event horizon into the black hole. The escaping piece of matter can have a greater mass-energy than the original infalling piece of matter. In other words, the captured piece has negative mass-energy. The Penrose process results in a decrease in the angular momentum of the black hole, and that reduction corresponds to a transference of energy whereby the momentum lost is converted to energy extracted. As a result of the Penrose process a rotating black hole can eventually lose all of its angular momentum, becoming a non-rotating (i.e. the Schwarzschild) black hole.

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