Lecture 3

III. PHYSICAL GEOMETRY OF SPACE-TIME

A. Proper time

One of the most central problems in the geometry of 4-spacetime can be formulated as follows. If the metric tensor is given, how is actual (measurable) time and distances related with coordinates x^0, x^1, x^2, x^3 chosen in arbitrary way? Let us consider the world line of an observer who uses some clock to measure the actual or proper time, $d\tau$, between two infinitesimally close events in the same place in space. How $d\tau$ is related to coordinate time dx^0. Obviously we should put in the interval

$$dx^1 = dx^2 = dx^3. \tag{III.1}$$

Let us define proper time exactly as in Special Relativity:

$$d\tau = \frac{ds}{c}, \tag{III.2}$$

then we have

$$ds^2 \equiv c^2 d\tau^2 = g_{ik} dx^i dx^k = g_{00}(dx^0)^2, \tag{III.3}$$

thus

$$d\tau = \frac{1}{c}\sqrt{g_{00}}dx^0. \tag{III.4}$$

For the proper time between any two events which are not necessary infinitesimally close occurring at the same point in space we have

$$\tau = \frac{1}{c}\int \sqrt{g_{00}}dx^0. \tag{III.5}$$

B. Physical distance

Separating the space and time coordinates in ds we have

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta + 2g_{0\alpha}dx^0 dx^\alpha + g_{00}(dx^0)^2. \tag{III.1}$$

To define dl we will use a light signal according to the following procedure: From some point B with spatial coordinates $x^\alpha + dx^\alpha$ a light signal emitted at the moment corresponding to time coordinate $x^0 + dx^{0(1)}$ propagates to a point A with spatial coordinates x^α and then after reflection at the moment corresponding to time coordinate x^0 the signal propagates back over the same path and is detected in the point B at the moment corresponding to time coordinate $x^0 + dx^{0(2)}$ as shown on Fig.3.1.
According to both Special and General Relativity the interval between any two events which belong to the same world line of light is always equal to zero:

$$ds = 0. \tag{III.2}$$

Solving this equation with respect to dx^0 we find two roots:

$$dx^{0(1)} = \frac{1}{g_{00}}\left(-g_{0\alpha}dx^\alpha - \sqrt{(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00})dx^\alpha dx^\beta}\right)$$

$$dx^{0(2)} = \frac{1}{g_{00}}\left(-g_{0\alpha}dx^\alpha + \sqrt{(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00})dx^\alpha dx^\beta}\right)$$

$$dx^{0(2)} - dx^{0(1)} = \frac{2}{g_{00}}\sqrt{(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00})dx^\alpha dx^\beta}. \tag{III.3}$$

Then

$$dl = \frac{c}{2}d\tau = \frac{c}{2}\frac{\sqrt{g_{00}}}{c}(dx^{0(2)} - dx^{0(1)}) \tag{III.4}$$

and finally

$$dl^2 = \gamma_{\alpha\beta}dx^\alpha dx^\beta, \text{ where } \gamma_{\alpha\beta} = -g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}}. \tag{III.5}$$

C. Synchronization of clocks

If we want to determine the distance between two not infinitesimally closed points, but points separated by some finite distance we should take an integral $\int dl$ along some path connecting the two points. Obviously, we should take dl over the path at the simultaneous moment of time. Hence, we should first to define what are simultaneous events and then we should synchronize clocks (again with using light signals) over finite volume in space along the path of integration.
The moment at the point B, corresponding to the time coordinate $x^0 + \Delta x^0$, is simultaneous to the moment at the point A, corresponding to the time coordinate x^0, if

$$x^0 + \Delta x^0 = x^0 + \frac{1}{2}(dx^{0(2)} + dx^{0(1)}), \tag{III.1}$$

i.e. the reading of clock in B is halfway between the moments of departure and return of the signal to that point, hence

$$\Delta x^0 = -\frac{g_{0\alpha}}{g_{00}}dx^\alpha. \tag{III.2}$$

As we are able now to define simultaneous events along any open curve, however, synchronization of clocks along a closed contour is impossible in general, since

$$-\oint \frac{g_{0\alpha}}{g_{00}}dx^\alpha \neq 0, \tag{III.3}$$

which means that starting synchronization in some point we return back with

$$\Delta x^0 \neq 0. \tag{III.4}$$

In other words, in an arbitrary reference system the synchronization of clocks in a whole space-time is impossible, but this is not the property of the space-time itself, but the property of the given frame of reference. We always can choose such a frame of reference in which all

$$g_{0\alpha} = 0 \tag{III.5}$$

and hence the synchronization of clocks in a whole space-time is possible. For that we should write 3 equations for 4 arbitrary functions, which is always possible.

D. Invariant 4-volume

To derive EFEs we should be able to calculate integrals over the all space and over the time coordinate

$$S_g = \int G d\tilde{\Omega}, \tag{III.1}$$

where $d\tilde{\Omega}$ is invariant, i.e. not depending on the frame of reference, the element of 4-volume and G is some scalar function. Thus we should understand what the invariant volume is.
Let us prove that the invariant volume is

$$d\tilde{\Omega} = \sqrt{-g} d\Omega, \tag{III.2}$$

where

$$d\Omega = dx^0 dx^1 dx^2 dx^3 \tag{III.3}$$

and g is the determinant of the metric tensor.
Proof
Let us introduce the Jacobian, J, of the transformation from the Galilean (locally inertial) frame of reference, (x'^0, x'^1, x'^2, x'^3), to the curvilinear coordinates (x^0, x^1, x^2, x^3)

$$J = \frac{\partial(x^0, x^1, x^2, x^3)}{\partial(x'^0, x'^1, x'^2, x'^3)} = |\frac{\partial x^i}{\partial(x'n}| = |S^i_n|, \tag{III.4}$$

where $|A^i_n|$ means the determinant of a matrix A^i_n. Then let us write the formula for the transformation of the contravariant metric tensor

$$g^{ik} = S^i_l S^k_m g^{lm(0)} = S^i_l S^k_m \eta^{lm}, \tag{III.5}$$

where

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{III.6}$$

Taking into account that the determinant of the reciprocal tensor g^{ik} is the inverse of the determinant of the tensor g_{ik}, we have

$$|g^{ik}| = \frac{1}{|g_{ik}|} = \frac{1}{g}. \tag{III.7}$$

Taking into account that the determinant of the product of matrices is equal to the product of their determinants (the fact known from any textbook on Linear Algebra), we obtain

$$|g^{ik}| = |S^i_l| \times |S^k_m| \times |\eta^{lm}| = J \times J \times (-1) = -J^2, \tag{III.8}$$

hence

$$\frac{1}{g} = -J^2 \text{ and } J = \frac{1}{\sqrt{-g}}. \tag{III.9}$$

From the definition of J we have

$$d\Omega \equiv dx^0 dx^1 dx^2 dx^3 = J dx'^0 dx'^1 dx'^2 dx'^3 = \frac{1}{\sqrt{-g}} dx'^0 dx'^1 dx'^2 dx'^3 = \frac{1}{\sqrt{-g}} d\Omega', \tag{III.10}$$

hence in all curvilinear coordinates

$$\sqrt{-g} d\Omega = d\Omega', \ \text{thus} \ d\tilde{\Omega} = \sqrt{-g} d\Omega \tag{III.11}$$

is invariant 4-volume.

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