Lecture 2

II. TENSORS

A. The principle of covariance and tensors

The Principle of Covariance predetermines the mathematical structure of General Relativity: all equations should contain tensors only. By definition, tensors are objects which are transformed properly in the course of coordinate transformations from one frame of reference to another. Taking into account that non-inertial frames of reference in the 4-dimensional space-time correspond to curvilinear coordinates, it is necessary to develop four-dimensional differential geometry in arbitrary curvilinear coordinates.

B. Transformation of coordinates

Let us consider the transformation of coordinates from one frame of reference (x^0, x^1, x^2, x^3) to another, ($x^{'0}$, $x^{'1}$, $x^{'2}$, $x^{'3}$):

$$x^0 = f^0(x'^0, x'^1, x'^2, x'^3), \;\; x^1 = f^1(x'^0, x'^1, x'^2, x'^3), \;\; x^2 = f^2(x'^0, x'^1, x'^2, x'^3), \;\; x^3 = f^3(x'^0, x'^1, x'^2, x'^3). \quad \text{(II.1)}$$

Then

$$dx^i = \frac{\partial x^i}{\partial x'^k} dx'^k = S^i_k dx'^k, \;\; i, k = 0,\ 1,\ 2,\ 3, \;\; \text{where} \;\; S^i_k = \frac{\partial x^i}{\partial x'^k} \quad \text{(II.2)}$$

is a transformation matrix. Remember that all repeating indices mean summation, otherwise even such a basic transformation would be very ugly when written. To demonstrate that summation convention is really very useful, I will write, the first and the last time, the same transformation without using the summation convention:

$$\left\{\begin{array}{l} dx^0 = \frac{\partial x^0}{\partial x'^0}dx'^0 + \frac{\partial x^0}{\partial x'^1}dx'^1 + \frac{\partial x^0}{\partial x'^2}dx'^2 + \frac{\partial x^0}{\partial x'^3}dx'^3 = S^0_0 dx'^0 + S^0_1 dx'^1 + S^0_2 dx'^2 + S^0_3 dx'^3, \\ dx^1 = \frac{\partial x^1}{\partial x'^0}dx'^0 + \frac{\partial x^1}{\partial x'^1}dx'^1 + \frac{\partial x^1}{\partial x'^2}dx'^2 + \frac{\partial x^1}{\partial x'^3}dx'^3 = S^1_0 dx'^0 + S^1_1 dx'^1 + S^1_2 dx'^2 + S^1_3 dx'^3, \\ dx^2 = \frac{\partial x^2}{\partial x'^0}dx'^0 + \frac{\partial x^2}{\partial x'^1}dx'^1 + \frac{\partial x^2}{\partial x'^2}dx'^2 + \frac{\partial x^2}{\partial x'^3}dx'^3 = S^2_0 dx'^0 + S^2_1 dx'^1 + S^2_2 dx'^2 + S^2_3 dx'^3, \\ dx^3 = \frac{\partial x^3}{\partial x'^0}dx'^0 + \frac{\partial x^3}{\partial x'^1}dx'^1 + \frac{\partial x^3}{\partial x'^2}dx'^2 + \frac{\partial x^3}{\partial x'^3}dx'^3 = S^3_0 dx'^0 + S^3_1 dx'^1 + S^3_2 dx'^2 + S^3_3 dx'^3 \end{array}\right. \tag{II.3}$$

C. Contravariant and covariant tensors

Now we can give the definition of the Contravariant four-vector: The Contravariant four-vector is the combination of four quantities (components) A^i, which are transformed like differentials of coordinates:

$$A^i = S^i_k A'^k. \tag{II.4}$$

Let φ is scalar field, then

$$\frac{\partial \varphi}{\partial x^i} = \frac{\partial \varphi}{\partial x'^k}\frac{\partial x'^k}{\partial x^i} = \tilde{S}^k_i \frac{\partial \varphi}{\partial x'^k}, \tag{II.5}$$

where $\tilde{S}^k_i$ is another transformation matric. What is the relation of this matrix to the previous transformation matrix S^i_k? If we take product of these matrices, we obtain

$$S^i_n \tilde{S}^n_k = \frac{\partial x^i}{\partial x'^n}\frac{\partial x'^n}{\partial x^k} = \frac{\partial x^i}{\partial x^k} = \delta^i_k, \tag{II.6}$$

where δ^i_k is so called Kronneker symbol, which actually is nothing but the unit matrix:

$$\delta^i_k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{II.7}$$

In other words $\tilde{S}^i_k$ is inverse or reciprocal with respect to S^i_k.
Now we can give the definition of the Covariant four-vector: The Covariant four-vector is the combination of four quantities (components) A_i, which are transformed like components of the gradient of a scalar field:

$$A_i = \frac{\partial x'^k}{\partial x^i} A'_k. \tag{II.8}$$

Note, that for contravariant vectors we always use upper indices, which are called contravariant indices, while for covariant vectors we use low indices, which are called covariant indices. In General Relativity summation convention always means that one of two repeating indices should be contravariant and another should be covariant.For example,

$$A^i B_i = A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3 \tag{II.9}$$

is the scalar product.
There is no summation if both indices are, say, covariant, for example:

$$A_i B_i = \begin{array}{l} A_0 B_0, \ \text{if } i = 0, \\ A_1 B_1, \ \text{if } i = 1, \\ A_2 B_2, \ \text{if } i = 2, \\ A_3 B_3, \ \text{if } i = 3. \end{array} \tag{II.10}$$

Now we can generalize the definitions of vectors and introduce tensors entirely in terms of transformation laws.
Scalar, A, is the tensor of the 0 rank. It has only $4^0 = 1$ component and 0 number of indices. Transformation law is

$$A = A', \tag{II.11}$$

we see that transformation matrices appear in transformation law 0 times.
Contravariant and covariant vectors are tensors of the 1 rank. They have $4^1 = 4$ components and 1 index. Corresponding transformation laws are

$$A^i = S^i_n A'^n, \tag{II.12}$$

$$A_i = \tilde{S}^n_i A'_n, \tag{II.13}$$

we see only 1 transformation matrix in each transformation law.
Contravariant tensor of the 2 rank has $4^2 = 16$ components and 2 contravariant indices. Corresponding transformation law is

$$A^{ik} = S^i_n S^k_m A'^{nm}, \tag{II.14}$$

we see 2 transformation matrices in the transformation law.
Covariant tensor of the 2 rank has $4^2 = 16$ components and 2 covariant indices. Corresponding transformation law is

$$A_{ik} = \tilde{S}^n_i \tilde{S}^m_k A'_{nm}, \tag{II.15}$$

we see 2 transformation matrices in the transformation law.
Mixed tensor of the 2 rank has $4^2 = 16$ components and 2 indices, 1 contravariant and 1 covariant. Corresponding transformation law is

$$A^i_k = S^i_n \tilde{S}^m_k A'^n_m, \tag{II.16}$$

we see 2 transformation matrices in the transformation law.
Covariant tensor of the 3 rank has $4^3 = 64$ components and 3 covariant indices. Corresponding transformation law is...and so on.
The most general definition: Mixed tensor of the $N + M$ rank with N contravariant and M covariant indices, has $4^{N+M} = 2^{2(N+M)}$ components and $N + M$ indices. Corresponding transformation law is

$$A^{i_1\, i_2\, \dots\, i_N}_{k_1\, k_2\, \dots\, k_M} = S^{i_1}_{n_1} S^{i_2}_{n_2} \dots S^{i_N}_{n_N} \tilde{S}^{m_1}_{k_1} \tilde{S}^{m_2}_{k_2} \dots \tilde{S}^{m_M}_{k_M} A'^{n_1\, n_2\, \dots\, n_N}_{m_1 m_2 \dots m_M}, \tag{II.17}$$

we see $N + M$ transformation matrices in the transformation law.

D. Reciprocal tensors

Two tensors A_{ik} and B^{ik} are called reciprocal to each other if

$$A_{ik} B^{kl} = \delta^l_i. \tag{II.18}$$

Now we can introduce a contravariant metric tensor g^{ik} which is reciprocal to the covariant metric tensor g_{ik}:

$$g_{ik} g^{kl} = \delta^l_i. \tag{II.19}$$

With the help of the metric tensor and its reciprocal we can form contravariant tensors from covariant tensors and vice versa, for example:

$$A^i = g^{ik} A_k, \quad A_i = g_{ik} A^k, \tag{II.20}$$

in other words we can rise and descend indices as we like, like a kind of juggling with indices. We can say that contravariant, covariant and mixed tensors can be considered as different representations of the same geometrical object.
For the contravariant metric tensor itself we have very important representation in terms of the transformation matrix from locally inertial frame of reference (galilean frame) to an arbitrary non-inertial frame, let us denote it as $S^i_{(0)k}$. We know that in the galilean frame of reference

$$g^{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv \eta^{ik} \equiv \mathrm{diag}(1, -1, -1, -1), \tag{II.21}$$

hence

$$g^{ik} = S^i_{(0)n} S^k_{(0)m} \eta^{lm} = S^i_{(0)0} S^k_{(0)0} - S^i_{(0)1} S^k_{(0)1} - S^i_{(0)2} S^k_{(0)2} - S^i_{(0)3} S^k_{(0)3}. \tag{II.22}$$

This means that if we know the transformation law from the local galilean frame of reference to an arbitrary frame of reference, we know the metric at this arbitrary frame of reference and, hence, we know the gravitational field which is identical to geometry!

E. Examples

Example 1. Given that g_{ik} is a covariant tensor of the second rank and that

$$ds^2 = g_{ik}dx^i dx^k, \tag{II.23}$$

prove that ds is a scalar.

The proof:

$$ds^2 = g_{ik}dx^i dx^k = (\tilde{S}_i^n \tilde{S}_k^m g'_{nm})(S_p^i dx'^p)(S_w^k dx'^w) = (\tilde{S}_i^n S_p^i)(\tilde{S}_k^m S_w^k)(g'_{nm}dx'^p dx'^w) =$$

$$= \delta_p^n \delta_w^m (g'_{nm}dx'^p dx'^w) = g'_{pw}dx'^p dx'^w = g'_{ik}dx'^i dx'^k = ds'^2, \tag{II.24}$$

hence $ds = ds'$ which means that ds is a scalar.

Example 2. How many independent components in the metric tensor?

The answer:

First, we can prove that the metric tensor is symmetric, i.e.

$$g_{ik} = g_{ki}. \tag{II.25}$$

Indeed,

$$ds^2 = g_{ik}dx^i dx^k = \frac{1}{2}(g_{ik}dx^i dx^k + g_{ik}dx^i dx^k) = \frac{1}{2}(g_{ki}dx^k dx^i + g_{ik}dx^i dx^k) = \frac{1}{2}(g_{ki} + g_{ik})dx^i dx^k =$$

$$= \tilde{g}_{ik}dx^i dx^k, \tag{II.26}$$

where

$$\tilde{g}_{ik} = \frac{1}{2}(g_{ki} + g_{ik}), \tag{II.27}$$

which is obviously a symmetric one. Then we just drop " ˜ ". The end of proof. Now the answer is obvious: altogether we have 4×4 components, 4 components on the diagonal, $3 + 2 + 1 = 6$ components above the diagonal and $3 + 2 + 1 = 6$ components under the diagonal and we know that these components are equal to components above the diagonal. Thus the final answer is there are $4 + 6 = 10$ independent components.

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F. Course work 1

Q1.

a) Formulate the equivalence principle and explain what is the difference in interpretation of this principle in Newtonian theory and in General relativity.

b) Explain the similarity between an "actual" gravitational field and a non-inertial reference system. Give the definition of a locally Galilean coordinate system.

c) Explain why an "actual" gravitational field cannot be eliminated by any transformation of coordinates over all space-time.

d) Show that in a uniformly rotating system of coordinates $x^{'}$, $y^{'}$, $z^{'}$, such that

$$x = x^{'}\cos\Omega t - y^{'}\sin\Omega t,\quad y = x^{'}\sin\Omega t + y^{'}\cos\Omega t,\quad z = z^{'}, \tag{F.1}$$

the interval ds has the following form:

$$ds^2 = g_{ik}dx^i dx^k = g^{'}_{ik}dx^{'i}dx^{'k} = [c^2 - \Omega^2(x^{'2} + y^{'2})]dt^2 - dx^{'2} - dy^{'2} - dz^{'2} + 2\Omega y^{'}dx^{'}dt - 2\Omega x^{'}dy^{'}dt. \tag{F.2}$$

Q2.

a) Formulate the covariance principle and explain the relationship between this principle and the principle of equivalence.

b) Give the definition of a contravariant vector in terms of the transformation of curvilinear coordinates.

c) Give the definition of a covariant vector in terms of the transformation of curvilinear coordinates.

d) What is the mixed tensor of the second rank in terms of the transformation of curvilinear coordinates (you can assume that a mixed tensor of the second rank is transformed as a product of covariant and contrvariant vectors).

e) Explain why the principle of covariance implies that all physical equations should contain only tensors.

Q3

a) Prove that the metric tensor is symmetric. Give a rigorous proof that the interval is a scalar.

b) Give the definition of the reciprocal tensors of the second rank. What is the contravariant metric tensor g^{ik}.

c) Show that in an arbitrary non-inertial frame

$$g^{ik} = S^i_{(0)0}S^k_{(0)0} - S^i_{(0)1}S^k_{(0)1} - S^i_{(0)2}S^k_{(0)2} - S^i_{(0)3}S^k_{(0)3},$$

where $S^i_{(0)k}$ is the transformation matrix from locally inertial frame of reference (galilean frame) to this non-inertial frame.

d) Demonstrate how using the reciprocal contravariant metric tensor g^{ik} and the covariant metric tensor g_{ik} you can form contravariant tensor from covariant tensors and vice versa.

Q4.

a) In the local Galilean frame $x^i_{[G]}$ of reference a mixed tensor of the second rank, C^i_k has the only one non-vanishing component, $C^0_{0[G]} = 1$, and all other components are equal to zero. Write down all components of this mixed tensor in arbitrary frame of reference. Express your result in terms of transformation matrix.

b) In the non-rotating system of Cartesian coordinates $(x,\ y,\ z)$ the only non-vanishing component of some tensor A^i_k is $A^1_1 = 1$ and all other components vanish. Using coordinate transformation from Cartesian to the uniformly rotating cylindrical coordinates $(r,\ \theta,\ \phi)$

$$x = r\cos(\theta + \Omega t),\quad y = r\sin(\theta + \Omega t),\quad z = Z, \tag{F.3}$$

show that in the latter coordinates

$$A^{'1}_0 = -\frac{r\Omega}{2c}\sin 2(\theta + \Omega t). \tag{F.4}$$

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