8. Matrices

8.0 *A Matrix is an Array of Objects with Rows and Columns*

Examples:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

These are **2 × 2**, or ***n* × *n*** with ***n* = 2**, **SQUARE** arrays

Example:

$$\begin{pmatrix} 1 & 2 & 4 \\ 6 & 0 & 7 \end{pmatrix}$$

This is a **2 × 3**, or ***m* × *n*** with ***m* = 2, *n* = 3**, **RECTANGULAR** array

Examples:

$\mathbf{V} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ is a **1 × 3** matrix. **V** is a **ROW VECTOR.**

$\mathbf{W} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is a **3 × 1** matrix. **W** is a **COLUMN VECTOR.**

$\mathbf{B} = \begin{pmatrix} b_1 & b_2 & b_3 & \cdots & b_n \end{pmatrix}$ is a **1 × *n*** matrix. **B** is an ***n-dimensional*** row vector

$\mathbf{C} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{pmatrix}$ is an ***n* × 1** matrix. **B** is an ***n-dimensional*** column vector

- We will use Square Matrices and Row and Column Vectors.

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & c \end{pmatrix}$$

M is a 2 × 2 matrix
with ELEMENTS *a*, *b*, *c*, *d*.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & & \vdots \\ a_{31} & a_{32} & a_{33} & & \vdots \\ \vdots & & & & \vdots \\ a_{n1} & \cdots & \cdots & \cdots & a_{nn} \end{pmatrix} = (a_{ij})$$

A is an $n \times n$ matrix
with the element a_{ij}
in the i^{th} column
and j^{th} row

So in **M** above, $m_{11} = a$ $m_{12} = b$
$m_{21} = c$ $m_{22} = d$

8.1 MATRIX OPERATIONS.

- Matrix Addition

Let A = (a_{ij}) be an $n \times m$ matrix
Let B = (b_{ij}) be a $p \times q$ matrix

Addition **A + B** is defined **only if $n = p$, $m = q$**.
(A and B *same shape).*

Then A + B = ($a_{ij} + b_{ij}$)

$$= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} & & \vdots \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} & & \vdots \\ \vdots & & & & \vdots \\ a_{n1}+b_{n1} & \cdots & \cdots & \cdots & a_{nn}+b_{nn} \end{pmatrix} = (a_{ij}+b_{ij})$$

Examples:

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 9 & 12 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

o Matrix Equality

The equalty $\mathbf{A} = \mathbf{B}$ is true if and only if (*iff*)

$n = p$

$m = q$

$a_{ij} = b_{ij} \quad \forall\, i, j$

Examples: $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathbf{A} = \mathbf{B}$

because: $n = p = m = q = 2$

$a_{11} = 0 = b_{11}$

$a_{12} = 1 = b_{12}$

$a_{21} = 1 = b_{21}$

$a_{22} = 0 = b_{22}$

$$\mathbf{A} = \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \qquad \mathbf{A} \neq \mathbf{B}$$

because: $a_{22} = 2 \neq b_{22} = 0$

o Matrix Multiplication: Inner Product

Like vector dot product:

Consider a **row** vector **u** *dot* a **column** vector **v**:

$$\mathbf{u.v} = \begin{pmatrix} u_x & u_y & u_z \end{pmatrix} . \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = u_x v_x + u_y v_y + u_z v_z$$

That is, $\mathbf{u.v} = \sum_{i=x}^{z} u_i v_i$

Relabel **u** and **v** as matrices with matrix subscripts, as

$$\mathbf{U} = \begin{pmatrix} u_{1x} & u_{1y} & u_{1z} \end{pmatrix} \qquad \mathbf{V} = \begin{pmatrix} v_{x1} \\ v_{y1} \\ v_{z1} \end{pmatrix}$$

$\mathbf{W} = \mathbf{U.V} = (w_{ij})$

$$w_{ij} = \sum_{p=x}^{z} u_{ip} v_{pi} \qquad i = j = 1$$

$$\mathbf{W} = (w_{11}) = u_{11}v_{11} + u_{12}v_{21} + u_{13}v_{31}$$

INNER PRODUCT because inner subscript *p* is summed over.

Generalising:

Let $\mathbf{A} = (a_{ij})$, an $n \times m$ matrix
Let $\mathbf{B} = (b_{ij})$, a $p \times q$ matrix

$\mathbf{C} = \mathbf{AB}$ exists if and only if $m = p$

Then $c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$

We prefer $c_{ij} = a_{ik} b_{kj}$

Convention – Einstein convention – of summing over repeated indices

Examples:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1\times5+2\times7 & 1\times6+2\times8 \\ 3\times5+4\times7 & 3\times6+4\times8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 5\times1+6\times3 & 5\times2+6\times4 \\ 7\times1+8\times3 & 7\times2+8\times4 \end{pmatrix} = \begin{pmatrix} 23 & 34 \\ 31 & 38 \end{pmatrix}$$

$\mathbf{AB} \neq \mathbf{BA}$

Examples:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

$$\begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{pmatrix}$$

Again, $\mathbf{AB} \neq \mathbf{BA}$

Examples:

$$\begin{pmatrix}1 & 3\\ 7 & 4\end{pmatrix}\begin{pmatrix}2\\ 5\end{pmatrix}=\begin{pmatrix}1\times 2+3\times 5\\ 7\times 2+4\times 5\end{pmatrix}=\begin{pmatrix}17\\ 34\end{pmatrix}=17\begin{pmatrix}1\\ 2\end{pmatrix}$$

$$\begin{pmatrix}a & b\end{pmatrix}\begin{pmatrix}6 & 7\\ 5 & 4\end{pmatrix}=\begin{pmatrix}6a+5b & 7a+4b\end{pmatrix}$$

$$\begin{pmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}a+2c & b+2d\\ 3a+4c & 3b+4d\\ 5a+6c & 5b+6d\end{pmatrix}$$

$$\begin{pmatrix}a & b\end{pmatrix}\begin{pmatrix}1\\ 2\end{pmatrix}=\begin{pmatrix}a+2b\end{pmatrix}$$

a 1 × 1 matrix

$1\times n$ times $n\times 1\rightarrow 1\times 1$

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}e & f\end{pmatrix}$$

is undefined

2×2 times 1×2

not the same

o Application of Matrix Multiplication: Rotation

Rotate a vector **v** clockwise through angle θ to become **v′**

We have $\mathbf{v}=(v_x,v_y)$, $\mathbf{v}'=(v'_x,v'_y)$

with $v'_x=\cos\theta\, v_x+\sin\theta\, v_y$, $v'_y=-\sin\theta\, v_x+\cos\theta\, v_y$,

Using matrices, this becomes,

$$\begin{pmatrix}v'_x\\ v'_y\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}v_x\\ v_y\end{pmatrix}$$

Or, **v′ = RV**

o Determinants

Length of rotated vector:

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2}$$

$$|\mathbf{v}'| = \sqrt{\begin{matrix}\cos^2\theta\, v_x^2 + \sin^2\theta\, v_x^2 + 2\cos\theta\sin\theta\, v_x v_y \\ + \sin^2\theta\, v_x^2 + \cos^2\theta\, v_x^2 - 2\cos\theta\sin\theta\, v_x v_y\end{matrix}}$$

$$= \sqrt{(\cos^2\theta + \sin^2\theta)v_x^2 + (\cos^2\theta + \sin^2\theta)v_y^2}$$

$$= \sqrt{v_x^2 + v_y^2} = |\mathbf{v}|$$ Unchanged, as expected

$\cos^2\theta + \sin^2\theta$ **is** $r_{11}r_{22} - r_{12}r_{21}$

This is the DETERMINANT of R, $\mathrm{Det}(\mathbf{R}) = |\mathbf{R}| = |(r_{ij})|$

It is a measure of the "size" of a matrix.

o Some Properties of Matrix Multiplication

- **(A + B) C = AC + AB**
- **(AB) C = A (BC)**
- **AB ≠ BA** in general – Non-Commutativity

8.2 MATRIX DEFINITIONS.

o Some Special Matrices

1. **NULL Matrix O** = (o_{ij}), $o_{ij} = 0$. **All elements are zero.**
2. **IDENTITY or UNIT Matrix I = E** (e_{ij}), $n \times n$. $e_{ij} = \delta_{ij}$.

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & \vdots \\ 0 & 0 & 1 & & \vdots \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 \end{pmatrix}$$

One's on diagonal,
Zeroes elsewhere

Example: $\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **is the** 3×3 **Identity Matrix**

Property: **AE = EA = A** $\forall$**A**

Example:

$$\mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \quad \mathbf{AE} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \quad \mathbf{EA} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

3. **DIAGONAL Matrix** $\mathbf{D} = (d_{ij}),\ d_{ij} = d_{ii}\delta_{ij}$.

$$\mathbf{D} = \begin{pmatrix} d_{11} & 0 & 0 & \cdots & 0 \\ 0 & d_{22} & 0 & & \vdots \\ 0 & 0 & d_{33} & & \vdots \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & d_{nn} \end{pmatrix}$$

Example: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ **are** Diagonal **Matrices**

4. **TRIANGULAR Matrices** $\mathbf{T} = (t_{ij}),\ t_{ij} = 0$ **for** $i > j$.

(i) Upper triangular matrices, $\mathbf{T} = (t_{ij}),\ t_{ij} = 0$ **for** $i > j$.

Example: $\mathbf{T} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix}$

(ii) Lower triangular matrices, $\mathbf{T} = (t_{ij}),\ t_{ij} = 0$ **for** $i < j$.

Example: $\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 10 \end{pmatrix}$

5. **SYMMETRIC Matrices** $\mathbf{S} = (s_{ij}),\ s_{ij} = s_{ji}$.

Example: $$\mathbf{S} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 7 \\ 3 & 6 & 8 & 9 \\ 4 & 7 & 9 & 10 \end{pmatrix}$$

6. **ANTISYMMETRIC Matrices** $\mathbf{A} = (a_{ij}),\ a_{ij} = -a_{ji}$.

Example: $$\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ -1 & 0 & 4 & 5 \\ -2 & -4 & 0 & 6 \\ -3 & -5 & -6 & 0 \end{pmatrix}$$

Note $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$

Zeroes on diagonal

o Matrix Transpose

$$\mathbf{A} = (a_{ij}) = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{1n} \\ a_{12} & a_{22} & & \vdots \\ \vdots & & & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{pmatrix}$$

is $m \times n$

The TRANSPOSE of **A** is $\mathbf{A}^{\mathrm{T}}$, the matrix obtained by swapping rows and columns:

$$\mathbf{A}^{\mathrm{T}} = (a_{ji}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{m1} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & & \vdots \\ a_{1n} & \cdots & \cdots & a_{nm} \end{pmatrix}$$

is $n \times m$

Example: $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\quad \mathbf{A}^{\mathrm{T}} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

Example: $\mathbf{A} = \begin{pmatrix} 3 & 1 \end{pmatrix}$ $\quad \mathbf{A}^{\mathrm{T}} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Example: $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad \mathbf{A}^{\mathrm{T}} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

o Complex Conjugate of a Matrix

If $\mathbf{A} = (a_{ij})$, then the COMPLEX CONJUGATE is $\mathbf{A}^* = (a_{ij}^*)$

Example: $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\quad \mathbf{A}^* = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

Example: $\mathbf{A} = \begin{pmatrix} 1 & 2+i \\ i & 3 \end{pmatrix}$ $\quad \mathbf{A}^* = \begin{pmatrix} 1 & 2-i \\ -i & 3 \end{pmatrix}$

Example: $\mathbf{A} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ $\quad \mathbf{A}^* = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$

o Hermitian Conjugate of a Matrix

†

If $\mathbf{A} = (a_{ij})$, then the HERMITIAN CONJUGATE is

$$\mathbf{A}^\dagger = (\mathbf{A}^*)^{\mathrm{T}} = (\mathbf{A}^{\mathrm{T}})^* = (a_{ji}^*)$$

Transposition + complex congugation.

Example: $\mathbf{A} = \begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix}$ $\quad \mathbf{A}^\dagger = \begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix}$

When, as in this example, $\mathbf{A}^\dagger = \mathbf{A}$, **A** is HERMITIAN

o Inverse of a Square Matrix

Given $\mathbf{A} = (a_{ij})$, the INVERSE $\mathbf{A}^{-1}$ satisfies $\mathbf{A}\mathbf{A}^{-1} = \mathbf{E} = \mathbf{A}^{-1}\mathbf{A}$

Recall that **E** is the identity matrix, 1's on diagonal, 0's elswhere

Note the inverse doesn't always exist, but when it does, it is UNIQUE.

Example: $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \mathbf{A}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad$ $\mathbf{E} = \mathbf{E}^{-1}$

Example: $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ $\quad \mathbf{A}^{-1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix}$

Verify both
$\mathbf{A}\mathbf{A}^{-1} = \mathbf{E}$
$\mathbf{A}^{-1}\mathbf{A} = \mathbf{E}$

o Trace of a Square Matrix

†

Given $\mathbf{A} = (a_{ij})$, $n \times n$, the TRACE of **A,** $Tr(\mathbf{A})$ is the sum of the diagonal elements:

$$Tr(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}$$

Example: $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $Tr(\mathbf{A}) = 1 + 4 = 5$

Example: $\mathbf{A} = \begin{pmatrix} 6 & 3 & 0 \\ 0 & 5 & 11 \\ 99 & 0 & 7 \end{pmatrix}$ $Tr(\mathbf{A}) = 6 + 5 + 7 = 18$