# Harmonic Oscillator

**Classical HO**
Particle mass $m$; restoring force constant $K$; equation

$$m\frac{d^2x}{dt^2} = -Kx \tag{1}$$

or

$$m\frac{d^2x}{dt^2} + \omega^2 x = 0; \qquad \omega = \left(\frac{K}{m}\right)^{\frac{1}{2}} \tag{2}$$

Solution to this which has $x = 0$ at $t = 0$ is

$$x = A\sin\omega T \tag{3}$$

with frequency of oscilation $\nu = \frac{\omega}{2\pi}$; $\omega$ is called the angular frequency.

**Quantum HO**
Potential corresponding to force $-Kx$ is

$$V(x) = \frac{1}{2}Kx^2 \tag{4}$$

Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}Kx^2\psi(x) = E\psi(x) \tag{5}$$

Change to dimenionless variables

$$y = \left(\frac{mK}{\hbar^2}\right)^{\frac{1}{4}} x = \alpha x; \qquad \epsilon = \frac{2E}{\hbar}\left(\frac{m}{K}\right)^{\frac{1}{2}} = \frac{2E}{\hbar\omega} = \frac{2E}{h\nu} \tag{6}$$

giving

$$\frac{d^2\psi}{dy^2} - y^2\psi(y) = -\epsilon\psi(y) \tag{7}$$

**Complimentary solution**
First solve simpler equation

$$\frac{d^2\psi}{dy^2} - y^2\psi(y) = 0 \tag{8}$$

(can think of this as equation as $|y| \to \infty$). Gives

$$\psi(y) = A\exp\left(-\frac{1}{2}y^2\right) + B\exp\left(\frac{1}{2}y^2\right) \tag{9}$$

Boundary conditions for a localised problem give $B = 0$ so that $\psi \to 0$ as $|y| \to \infty$. Assume full solution of form

$$\psi(y) = H(y)\exp\left(-\frac{1}{2}y^2\right)$$

$$\frac{d\psi}{dy} = \frac{dH}{dy} \exp\left(-\frac{1}{2}y^2\right) - y\psi$$

$$\frac{d^2\psi}{dy^2} = \frac{d^2H}{dy^2} \exp\left(-\frac{1}{2}y^2\right) - y\frac{dH}{dy} \exp\left(-\frac{1}{2}y^2\right) - \psi - y\frac{dH}{dy} \exp\left(-\frac{1}{2}y^2\right) + y^2\psi \tag{10}$$

which gives

$$\frac{d^2\psi}{dy^2} - y^2\psi(y) = -\epsilon\psi(y) = \exp\left(-\frac{1}{2}y^2\right)\left[\frac{d^2H}{dy^2} - 2y\frac{dH}{dy} - H\right] \tag{11}$$

so the equation to solve is

$$\frac{d^2H}{dy^2} - 2y\frac{dH}{dy} + (\epsilon - 1)H = 0 \tag{12}$$

This equation has $p(y) = -2y$ and $q(y) = -(\epsilon - 1)$, so there are no singular points. So can obtain two simple series solution about $y = 0$, these will have radius of convegence, $\rho = \infty$. Also note that the equation is **even** so expect separate even and odd solutions

$$H(y) = \sum_{n=0}^{\infty} a_n y^n;$$

$$\frac{dH}{dy} = \sum_{n=0}^{\infty} n a_n y^{n-1};$$

$$\frac{d^2H}{dy^2} = \sum_{n=0}^{\infty} n(n-1) a_n y^{n-2} \tag{13}$$

so

$$\sum_{n=0}^{\infty} n(n-1)a_n y^{n-2} - 2\sum_{n=0}^{\infty} n a_n y^n - (\epsilon - 1)\sum_{n=0}^{\infty} a_n y^n = 0 \tag{14}$$

tidying this up and changing the dummy variable on the first sum by $n \to n+2$ gives

$$\sum_{n=-2}^{\infty} (n+1)(n+2)a_n y^n + \sum_{n=0}^{\infty} (\epsilon - 1 - 2n)a_n y^n = 0 \tag{15}$$

For this equation to be true for **all** values of $y$, the coefficient of each power of $y$ must be **separately** equated to zero. This gives

$$2a_2 + (\epsilon - 1)a_0 = 0 \qquad \text{coef. of } y^0;$$

$$a_{j+2}(j+2)(j+1) - [\epsilon - 1 - 2j]a_j = 0 \qquad \text{coef. of } y^j. \tag{16}$$

giving a recurrence relation

$$a_{j+2} = \frac{2j - \epsilon + 1}{(j+1)(j+2)} a_j \qquad j = 0, 1, 2, \ldots \tag{17}$$

The series must **terminate** otherwise $H(y)$ and hence $\psi(x)$ go as $\exp(y^2)$, ie as the solution already rejected. If highest power of $y$ in a solution is $y^n$, then $a_{n+1}$ and $a_{n+2}$ **must be zero**. This means

$$a_{n+2} = 0 = \frac{2n - \epsilon + 1}{(n+1)(n+2)} a_n \tag{18}$$

which gives

$$2n - \epsilon + 1 = 0 \tag{19}$$

or $\epsilon = 2n + 1$ as the physically allowed levels of the HO, which are

$$E = (n + \frac{1}{2})h\nu = (n + \frac{1}{2})\hbar\omega \qquad n = 0, 1, 2, \ldots \tag{20}$$

The polynomials $H(y)$ are called **Hermite Polynomials**, generally written $H_n(y)$. By convention they are written so that $a_n = 2^n$. They have recurrence relation

$$a_{j+2} = \frac{2(j - n)}{(j + 1)(j + 2)} a_j. \tag{21}$$

First few Hermite polynomials

$$\begin{aligned} H_0(y) &= 1, \\ H_1(y) &= 2y, \\ H_2(y) &= 4y^2 - 2, \end{aligned} \tag{22}$$

Normalisation constant:

$$N_n = \left( \frac{\alpha}{\pi^{\frac{1}{2}} 2^n n!} \right)^{\frac{1}{2}} \tag{23}$$