Proof of equation (40) from Lecture 18

$\underline{J}$ is the total angular momentum vector, and $\underline{V}$ is a vector operator.

The commutation relations for the components of $\underline{J}$ and $\underline{V}$ are

$$[J_x, V_x] = 0, \quad [J_x, V_y] = i\hbar V_z, \quad [J_x, V_z] = -i\hbar V_y \tag{1}$$
$$[J_y, V_x] = -i\hbar V_z, \quad [J_y, V_y] = 0, \quad [J_y, V_z] = i\hbar V_x \tag{2}$$
$$[J_z, V_x] = -i\hbar V_y, \quad [J_z, V_y] = -i\hbar V_x, \quad [J_z, V_z] = 0 \tag{3}$$

Also, the commutation relations for the individualt components of $\underline{J}$ are:

$$[J_x, J_y] = i\hbar J_z, \quad [J_y, J_z] = i\hbar J_x \quad [J_z, J_x] = i\hbar J_y \tag{4}$$

Using these we can show that:

$$\underline{J} \times \underline{V} + \underline{V} \times \underline{J} = 2i\hbar \underline{V} \tag{5}$$

and also that

$$\left[\underline{J}^2, \left[\underline{J}^2, \underline{V}\right]\right] = 2\hbar^2 \left(J^2 \underline{V} + \underline{V} J^2\right) - 4\hbar^2 \left(\underline{V} \cdot \underline{J}\right) \underline{J} \tag{6}$$

Now the matrix element $\langle\psi| \left[\underline{J}^2, \left[\underline{J}^2, \underline{V}\right]\right] |\psi\rangle$ of the left hand side of equation 6 for $|\psi\rangle = |lsjm_j\rangle$ is equal to zero, so we must have:

$$\langle lsjm_j| \left(J^2 \underline{V} + \underline{V} J^2\right) |lsjm_j\rangle = 2\langle lsjm_j| \left(\underline{V} \cdot \underline{J}\right) \underline{J} |lsjm_j\rangle \tag{7}$$

and so:

$$j(j+1)\hbar^2 \langle lsjm_j|\underline{V}|lsjm_j\rangle = \langle lsjm_j| \left(\underline{V} \cdot \underline{J}\right) \underline{J} |lsjm_j\rangle \tag{8}$$

Now recall the energy shift from the lecture:

$$\Delta E_S = \int \psi^* \frac{\mu_B}{\hbar} B \underline{\hat{z}} \cdot \underline{S} \psi d\tau \tag{9}$$

where we have put $\underline{B} = B\underline{z}$.

This energy is:

$$\Delta E_S = \frac{\mu_B}{\hbar} B \langle\psi|S_z|\psi\rangle \tag{10}$$

So we can use equation 8 by setting $\underline{V} = \underline{S}$ and taking the $z-$ component, which gives;

$$\begin{aligned}
\langle\psi|S_z|\psi\rangle &= \frac{1}{j(j+1)\hbar^2}\langle\psi|(\underline{S}\cdot\underline{J})J_z|\psi\rangle \\
&= \frac{m_j}{j(j+1)\hbar}\langle\psi|(\underline{S}\cdot\underline{J})|\psi\rangle
\end{aligned} \tag{11}$$

We can now write $\underline{L} = \underline{J} - \underline{S}$, so that $L^2 = J^2 + S^2 - 2\underline{S}\cdot\underline{J}$. And:

$$\begin{aligned}
\langle\psi|\underline{S}\cdot\underline{J}|\psi\rangle &= \frac{1}{2}\langle\psi|J^2 + S^2 - L^2|\psi\rangle \\
&= \frac{1}{2}\hbar^2\left(j(j+1) + s(s+1) - l(l+1)\right)
\end{aligned} \tag{12}$$

which by inserting into the above gives:

$$\langle\psi|S_z|\psi\rangle = m_j\hbar\left[\frac{j(j+1) + s(s+1) - l(l+1)}{2j(j+1)}\right] \tag{13}$$

And so the energy shift is:

$$\Delta E_S = m_j\mu_B B\left[\frac{j(j+1) + s(s+1) - l(l+1)}{2j(j+1)}\right] \tag{14}$$

as given in the lectures (equation 40).