1. Eigenvalues and Eigenvectors

To facilitate the study of eigenvalues and eigenvectors in depth, we need to consider complex matrices and vectors. Let

$$\mathbb{C}^n = \left\{ \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \,\middle|\, z_1, \ldots, z_n \in \mathbb{C} \right\}$$

be the complex vector space of $n \times 1$ complex matrices, in which vector addition and scalar multiplication are defined entry-wise.

Example 1.1. Let $\mathbf{z}, \mathbf{w} \in \mathbb{C}^3$ and $\alpha \in \mathbb{C}$, with

$$\mathbf{z} = \begin{pmatrix} 1+i \\ 2i \\ 3 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} -2+3i \\ 1 \\ 2+i \end{pmatrix}, \quad \alpha = (1+2i)\,.$$

Then

$$\mathbf{z}+\mathbf{w} = \begin{pmatrix} (1+i)+(-2+3i) \\ 2i+1 \\ 3+(2+i) \end{pmatrix} = \begin{pmatrix} -1+4i \\ 1+2i \\ 5+i \end{pmatrix}$$

$$\alpha\mathbf{z} = \begin{pmatrix} (1+2i)(1+i) \\ (1+2i)(2i) \\ (1+2i)\cdot 3 \end{pmatrix} = \begin{pmatrix} 1+2i+i+2i^2 \\ 2i+(2i)^2 \\ 3+6i \end{pmatrix} = \begin{pmatrix} -1+3i \\ -4+2i \\ 3+6i \end{pmatrix}$$

More generally, we denote by $\mathbb{C}^{m\times n}$ the complex vector space of $m \times n$ complex matrices, with the usual matrix addition and scalar multiplication. Needless to say, a real matrix A is also a complex matrix, in other words, we have

$$\mathbb{R}^{m\times n} \subset \mathbb{C}^{m\times n}.$$

Definition 1.2. An *eigenvector* of an $n \times n$ complex matrix $A \in \mathbb{C}^{n\times n}$ is a **nonzero** vector $\mathbf{x} \in \mathbb{C}^n$ such that

$$A\mathbf{x} = \lambda\mathbf{x}\,,$$

for some scalar λ. A scalar λ is called an *eigenvalue* of A if there is a **nonzero** vector $\mathbf{x}$ satisfying $A\mathbf{x} = \lambda\mathbf{x}$, in which case we say that $\mathbf{x}$ is an **eigenvector corresponding to the eigenvalue** λ.

Note 1. An eigenvector is always nonzero whereas an eigenvalue could be 0.

Note 2. An eigenvector x could be *real*, i.e. $\mathbf{x} \in \mathbb{R}^n$.

In the above definition, we say that A has a *real* eigenvalue if $\lambda \in \mathbb{R}$. Also, we say that A has a real eigenvector $\mathbf{v}$ if $\mathbf{v} \in \mathbb{R}^n$, that is, all entries in $\mathbf{v}$ are real.

Note that a *real* square matrix A may have *complex* eigenvalue! Recall that a real square matrix is called *symmetric* if $A^T = A$. A remarkable fact in linear algebra is that

All eigenvalues of a real symmetric matrix are REAL with a real eigenvector.

One of our goals is to establish this important fact.

We shall now investigate how to determine all the eigenvalues and eigenvectors of an $n \times n$ matrix A. We start by observing that the defining equation $A\mathbf{x} = \lambda\mathbf{x}$ can be written

$$(A - \lambda I)\mathbf{x} = \mathbf{0}\,. \tag{1}$$

Thus λ is an eigenvalue of A if and only if (1) has a non-zero solution. The set

$$\{\mathbf{x} : (A - \lambda I)\mathbf{x} = \mathbf{0}\}$$

of solutions of (1) is $N(A - \lambda I)$, that is, the nullspace of $A - \lambda I$. Thus, λ is an eigenvalue of A if and only if

$$N(A - \lambda I) \neq \{\mathbf{0}\}\ ,$$

and any **nonzero** vector in $N(A-\lambda I)$ is an eigenvector belonging to λ. Moreover, by the Invertible Matrix Theorem, (1) has a non-trivial solution if and only if the matrix $A - \lambda I$ is singular, or equivalently

$$\det(A - \lambda I) = 0\,. \tag{2}$$

Notice now that if the determinant in (2) is expanded we obtain a polynomial of degree n in the variable λ,

$$p(\lambda) = \det(A - \lambda I)\,,$$

called the **characteristic polynomial of** A , and equation (2) is called the **characteristic equation of** A. So, in other words, the roots of the characteristic polynomial of A are exactly the eigenvalues of A. The following theorem summarises our findings so far:

Theorem 1.3. *Let A be an $n \times n$ matrix and λ a scalar. The following statements are equivalent:*

(a) λ *is an eigenvalue of* A*;*
(b) $(A - \lambda I)\mathbf{x} = \mathbf{0}$ *has a non-trivial solution;*
(c) $N(A - \lambda I) \neq \{\mathbf{0}\}$*;*
(d) $A - \lambda I$ *is singular;*
(e) $\det(A - \lambda I) = 0$.

In view of the above theorem the following concept arises naturally:

Definition 1.4. If A is a square matrix and λ an eigenvalue of A, then $N(A-\lambda I)$ is called the **eigenspace corresponding to** λ and is denoted by

$$E(\lambda) = N(A - \lambda I) = \{\mathbf{x} : A\mathbf{x} = \lambda\mathbf{x}\}.$$

For a complex $n \times n$ matrix A, the eigenspace $E(\lambda)$ is a subspace of $\mathbb{C}^n$. If A is a real $n \times n$ matrix which has a *real* eigenvalue λ with *real* eigenvector, we can also consider the eigenspace

$$\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \lambda\mathbf{x}\}$$

which is a subspace of $\mathbb{R}^n$. We call it the **eigenspace corresponding to λ in $\mathbb{R}^n$** or simply, the **real** eigenspace corresponding to λ. If there is no confusion, we will also denote it by $E(\lambda)$.

Example 1.5. Let

$$A = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix}.$$

Find the eigenvalues and corresponding **real** eigenspaces.

Solution. A slightly tedious calculation using repeated cofactor expansions shows that the characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{vmatrix} = -\lambda(\lambda-1)^2,$$

so A has real eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 1$.

To find the real eigenspace corresponding to λ_1 we find the nullspace of $A - \lambda_1 I = A$ using Gaussian elimination:

$$A = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix} \sim \cdots \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

so setting $x_3 = \alpha$ we find $x_2 = 0 - (-1)x_3 = \alpha$ and $x_1 = 0 - (-1)x_3 = \alpha$. Thus, every vector in $N(A)$ is of the form

$$\begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

so the real eigenspace corresponding to the eigenvalue 0 is

$$E(0) = \left\{ \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\}.$$

To find the eigenspace corresponding to λ_2 we find the nullspace of $A - \lambda_2 I = A - I$, again using Gaussian elimination:

$$A - I = \begin{pmatrix} 2-1 & -3 & 1 \\ 1 & -2-1 & 1 \\ 1 & -3 & 2-1 \end{pmatrix} = \begin{pmatrix} 1 & -3 & 1 \\ 1 & -3 & 1 \\ 1 & -3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so setting $x_2 = \alpha$ and $x_3 = \beta$ we find $x_1 = 3x_2 - x_3 = 3\alpha - \beta$. Thus every vector in $N(A - I)$ is of the form

$$\begin{pmatrix} 3\alpha - \beta \\ \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},$$

and so the eigenspace corresponding to the eigenvalue 1 is

$$E(1) = \left\{ \alpha \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \,\middle|\, \alpha, \beta \in \mathbb{R} \right\}.$$

□

Example 1.6. Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}.$$

Solution. Using the fact that the determinant of a triangular matrix is the product of the diagonal entries we find

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda)(6-\lambda),$$

so the eigenvalues of A are 1, 4, and 6. □

The above example and its method of solution are easily generalised:

Theorem 1.7. *The eigenvalues of a triangular matrix are precisely the diagonal entries of the matrix.*

The next theorem gives an important sufficient (but not necessary) condition for two matrices to have the same eigenvalues. It also serves as the foundation for many numerical procedures to approximate eigenvalues of matrices, some of which you will encounter if you take the module MTH5110, Introduction to Numerical Computing.

Theorem 1.8. *Let A and B be two $n \times n$ matrices and suppose that A and B are similar, that is, there is an invertible matrix $S \in \mathbb{R}^{n\times n}$ such that $B = S^{-1}AS$. Then A and B have the same characteristic polynomial, and, consequently, have the same eigenvalues.*

Proof. If $B = S^{-1}AS$, then

$$B - \lambda I = S^{-1}AS - \lambda I = S^{-1}AS - \lambda S^{-1}S = S^{-1}(AS - \lambda S) = S^{-1}(A - \lambda I)S.$$

Thus, using the multiplicativity of determinants,

$$\det(B - \lambda I) = \det(S^{-1})\det(A - \lambda I)\det(S) = \det(A - \lambda I),$$

because $\det(S^{-1})\det(S) = \det(S^{-1}S) = \det(I) = 1$. □

Let V be a real or complex vector space and let $L : V \to V$ be a linear map.

A scalar λ is called an eigenvalue of L if there is a **nonzero** vector $\mathbf{v} \in V$ such that $L(\mathbf{v}) = \lambda\mathbf{v}$, in which case $\mathbf{v}$ is called an **eigenvector corresponding to the eigenvalue** λ.

A vector $\mathbf{v} \in V$ is called an *eigenvector* of L if $\mathbf{v} \neq \mathbf{0}$ and there is a scalar λ such that $L(\mathbf{v}) = \lambda\mathbf{v}$.

Given $A \in \mathbb{R}^{n\times n}$ and the linear map $L_A : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$L_A(\mathbf{x}) = A\mathbf{x},$$

we have $L_A(\mathbf{x}) = \lambda\mathbf{x}$ if and only if $A\mathbf{x} = \lambda\mathbf{x}$. Therefore L_A and A have the same eigenvalues and eigenvectors.

Likewise, given a complex $n\times n$ matrix, we can define a linear map $L_A : \mathbb{C}^n \to \mathbb{C}^n$ by

$$L_A(\mathbf{x}) = A\mathbf{x} \qquad (\mathbf{x} \in \mathbb{C}^n).$$

In this case, both A and L_A have the same eigenvalues in $\mathbb{C}$ and eigenvectors in $\mathbb{C}^n$.

2. Diagonalisation

We recall:

Definition 2.1. Let A and B be two $n \times n$ matrices. The matrix B is said to be **similar to** A if there is an invertible $S \in \mathbb{R}^{n\times n}$ such that

$$B = S^{-1}AS\,.$$

Notice that if B is similar to A, then A is similar to B, because if $R = S^{-1}$, then

$$A = SBS^{-1} = R^{-1}BR\,.$$

Thus we may simply say that A and B are similar matrices.

Definition 2.2. An $n \times n$ matrix A is said to be **diagonalisable** if it is similar to a diagonal matrix, that is, if there is an invertible matrix $P \in \mathbb{R}^{n\times n}$ such that

$$P^{-1}AP = D\,,$$

where D is a diagonal matrix. In this case we say that P **diagonalises** A.

Note that if A is a matrix which is diagonalised by P, that is, $P^{-1}AP = D$ with D diagonal, then

$$\begin{aligned} A &= PDP^{-1}\,, \\ A^2 &= PDP^{-1}PDP^{-1} = PD^2P^{-1}\,, \\ A^3 &= AA^2 = PDP^{-1}PD^2P^{-1} = PD^3P^{-1}\,, \end{aligned}$$

and in general

$$A^k = PD^kP^{-1}\,,$$

for any $k \geq 1$. Thus powers of A are easily computed, as claimed.

Theorem 2.3 (Diagonalisation Theorem). *An $n \times n$ matrix A is diagonalisable if and only if A has n linearly independent eigenvectors.*

Proof. $\Leftarrow$: Let A have n linearly independent eigenvectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ with corresponding eigenvalues $\lambda_1, \cdots, \lambda_n$. Let P be the $n \times n$ matrix with columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Since $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are linearly independent, P is invertible.

Let D be the diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$. Then

$$AP = A\begin{pmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_n\end{pmatrix} = \begin{pmatrix}A\mathbf{v}_1 & \cdots & A\mathbf{v}_n\end{pmatrix} = \begin{pmatrix}\lambda\mathbf{v}_1 & \cdots & \lambda_n\mathbf{v}_n\end{pmatrix} \tag{3}$$

$$= \begin{pmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_n\end{pmatrix}\begin{pmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{pmatrix} = PD$$

which gives $A = P^{-1}DP$, that is, A is diagonalisable.

$\Rightarrow$. We reverse the above proof. Suppose that A is diagonalisable with $P^{-1}AP = D$, where the invertible matrix P has columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$, and D is a diagonal matrix with diagonals $\lambda_1, \cdots, \lambda_n$. Then $AP = PD$ gives

$$\begin{pmatrix}A\mathbf{v}_1 & \cdots & A\mathbf{v}_n\end{pmatrix} = \begin{pmatrix}\lambda_1\mathbf{v}_1 & \cdots & \lambda_n A\mathbf{v}_n\end{pmatrix} . \tag{4}$$

Thus

$$A\mathbf{v}_1 = \lambda_1\mathbf{v}_1\,, \quad A\mathbf{v}_2 = \lambda_2\mathbf{v}_2\,, \quad \ldots, \quad A\mathbf{v}_n = \lambda_n\mathbf{v}_n\,. \tag{5}$$

Since P is invertible its columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$ must be linearly independent. Moreover, none of its columns can be zero, so (5) implies that $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A with corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. □

Example 2.4. Diagonalise the following matrix, if possible:

$$A = \begin{pmatrix}-7 & 3 & -3 \\ -9 & 5 & -3 \\ 9 & -3 & 5\end{pmatrix} .$$

Solution. A slightly tedious calculation shows that the characteristic polynomial is given by

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix}-7-\lambda & 3 & -3 \\ -9 & 5-\lambda & -3 \\ 9 & -3 & 5-\lambda\end{vmatrix} = -\lambda^3 + 3\lambda^2 - 4\,.$$

The cubic p above can be factored by spotting that -1 is a root. Polynomial division then yields

$$p(\lambda) = -(\lambda+1)(\lambda^2 - 4\lambda + 4) = -(\lambda+1)(\lambda-2)^2\,,$$

so the distinct eigenvalues of A are 2 and -1.

The usual method now produces a basis for each of the two eigenspaces and it turns out that

$$N(A-2I) = \operatorname{Span}(\mathbf{v}_1, \mathbf{v}_2), \quad \text{where} \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix},$$

$$N(A+I) = \operatorname{Span}(\mathbf{v}_3), \quad \text{where} \quad \mathbf{v}_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

You may now want to confirm, using your favourite method, that the three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent. In fact, this is not really necessary: the union of basis vectors for eigenspaces always produces linearly independent vectors.

Thus, A is diagonalisable, since it has 3 linearly independent eigenvectors. In order to find the diagonalising matrix P we recall that defining

$$P = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ 3 & 0 & -1 \\ 0 & 3 & 1 \end{pmatrix}$$

does the trick, that is, $P^{-1}AP = D$, where D is the diagonal matrix whose entries are the eigenvalues of A and where the order of the eigenvalues matches the order chosen for the eigenvectors in P, that is,

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

It is good practice to check that P and D really do the job they are supposed to do:

$$AP = \begin{pmatrix} -7 & 3 & -3 \\ -9 & 5 & -3 \\ 9 & -3 & 5 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ 3 & 0 & -1 \\ 0 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 1 \\ 6 & 0 & 1 \\ 0 & 6 & -1 \end{pmatrix},$$

$$PD = \begin{pmatrix} 1 & -1 & -1 \\ 3 & 0 & -1 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 1 \\ 6 & 0 & 1 \\ 0 & 6 & -1 \end{pmatrix},$$

so $AP = PD$, and hence $P^{-1}AP = D$ as required. □

Example 2.5. Diagonalise the following matrix, if possible:

$$A = \begin{pmatrix} -6 & 3 & -2 \\ -7 & 5 & -1 \\ 8 & -3 & 4 \end{pmatrix}.$$

Solution. The characteristic polynomial of A turns out to be exactly the same as in the previous example:

$$\det(A - \lambda I) = -\lambda^3 + 3\lambda^2 - 4 = -(\lambda + 1)(\lambda - 2)^2 .$$

Thus the eigenvalues of A are 2 and -1. However, in this case it turns out that both eigenspaces are 1-dimensional:

$$N(A - 2I) = \mathrm{Span}\,(\mathbf{v}_1) \quad \text{where} \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} ,$$

$$N(A + I) = \mathrm{Span}\,(\mathbf{v}_2) \quad \text{where} \quad \mathbf{v}_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} .$$

Since A has only 2 linearly independent eigenvectors, the Diagonalisation Theorem implies that A is not diagonalisable. □

Put differently, the Diagonalisation Theorem states that a matrix $A \in \mathbb{R}^{n\times n}$ is diagonalisable if and only if A has enough eigenvectors to form a basis of $\mathbb{R}^n$.

Theorem 2.6. *If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are linearly independent.*

Proof. By contradiction. Suppose to the contrary that the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are linearly dependent. We may then assume (after reordering the $\mathbf{v}$'s and the λ's if necessary), that there is an index $p < r$ such that $\mathbf{v}_{p+1}$ is a linear combination of the preceding *linearly independent* vectors. Thus there exist scalars $c_1, \ldots, c_p$ such that

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{v}_{p+1} . \tag{6}$$

Multiplying both sides of the above equation by A and using the fact that $A\mathbf{v}_k = \lambda_k\mathbf{v}_k$ for each k, we obtain

$$c_1\lambda_1\mathbf{v}_1 + \cdots + c_p\lambda_p\mathbf{v}_p = \lambda_{p+1}\mathbf{v}_{p+1} . \tag{7}$$

Multiplying both sides of (6) by λ_{p+1} and subtracting the result from (7), we see that

$$c_1(\lambda_1 - \lambda_{p+1})\mathbf{v}_1 + \cdots + c_p(\lambda_p - \lambda_{p+1})\mathbf{v}_p = \mathbf{0} . \tag{8}$$

Since the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are linearly independent, the coefficients in (8) must all be zero. But none of the factors $\lambda_k - \lambda_{p+1}$ is zero, because the eigenvalues are distinct, so we must have $c_1 = \ldots = c_p = 0$. But then (6) implies that $\mathbf{v}_{p+1} = \mathbf{0}$, which is impossible, because $\mathbf{v}_{p+1}$ is an eigenvector. Thus our assumption that the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are linearly dependent must be false, that is, the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are linearly independent. □

A useful special case of the Diagonalisation Theorem is the following:

Theorem 2.7. *An $n \times n$ matrix with n distinct eigenvalues is diagonalisable.*

Proof. Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be eigenvectors corresponding to the n distinct eigenvalues of A. Then the n vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly independent by Theorem 2.6. Hence A is diagonalisable by the Diagonalisation Theorem. □

Remark 2.8. Note that the above condition for diagonalisability is *sufficient* but not *necessary*: an $n \times n$ matrix which does not have n distinct eigenvalues may or may not be diagonalisable (see Examples 2.4 and 2.5).

Example 2.9. The matrix

$$A = \begin{pmatrix} 1 & -1 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{pmatrix}$$

is diagonalisable, since it has three distinct eigenvalues 1, 2, and 3.

3. Spectral Theorem

We are going to establish one of the great theorems in linear algebra, namely, the Spectral Theorem, which says that every real symmetric matrix can be diagonalised by an orthogonal matrix. Strangely, to prove this result concerning *real* matrices, we take a route through complex matrices and vectors.

[Spectral Theorem for Symmetric Matrices]

Let $A \in \mathbb{R}^{n \times n}$ be symmetric. Then there is an orthogonal matrix $Q \in \mathbb{R}^{n \times n}$ such that

$$Q^T A Q = D,$$

where $D \in \mathbb{R}^{n \times n}$ is diagonal.

In other words, every real symmetric matrix can be diagonalised by an orthogonal matrix.

Definition 3.1. A real square matrix A is called *orthogonal* if $AA^T = I$, in which case, we also have $A^T A = I$.

To achieve our goal, we need to recall the following basic fact.

Theorem 3.2 (Fundamental Theorem of Algebra)**.** *If p is a complex polynomial of degree $n \geq 1$, that is,*

$$p(z) = c_n z^n + \cdots + c_1 z + c_0 \,,$$

where $c_0, c_1, \dots, c_n \in \mathbb{C}$, then p has at least one (possibly complex) root.

Corollary 3.3. *Every matrix $A \in \mathbb{C}^{n \times n}$ has at least one (possibly complex) eigenvalue and a corresponding eigenvector $\mathbf{z} \in \mathbb{C}^n$.*

Proof. Since λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$ and since $p(\lambda) = \det(A - \lambda I)$ is a polynomial with complex coefficients of degree n, the assertion follows from the Fundamental Theorem of Algebra. □

Lemma 3.4. *Let $A \in \mathbb{R}^{n\times n}$ be symmetric. Then, for any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,*

$$\langle A\mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, A\mathbf{y}\rangle .$$

Proof. Recall that $\langle \mathbf{z}, \mathbf{w}\rangle = \mathbf{z}\cdot\mathbf{w} = \mathbf{z}^T\mathbf{w}$ for $\mathbf{z}, \mathbf{w} \in \mathbb{R}^n$. Since $A^T = A$, we have

$$\langle A\mathbf{x}, \mathbf{y}\rangle = (A\mathbf{x})\cdot\mathbf{y} = (A\mathbf{x})^T\mathbf{y} = \mathbf{x}^T A^T\mathbf{y} = \mathbf{x}^T A\mathbf{y} = \mathbf{x}\cdot(A\mathbf{y}) = \langle \mathbf{x}, A\mathbf{y}\rangle .$$

□

The following lemma contains the key result that will allow us to prove the Spectral Theorem.

Lemma 3.5. *Every symmetric matrix $A \in \mathbb{R}^{n\times n}$ has at least one real eigenvalue with corresponding real eigenvector $\mathbf{v} \in \mathbb{R}^n$.*

Proof. By Corollary 3.3 we know that A has at least one complex eigenvalue λ with corresponding eigenvector $\mathbf{z} \in \mathbb{C}^n$, that is

$$A\mathbf{z} = \lambda\mathbf{z}\,. \tag{9}$$

Write

$$\begin{aligned} \lambda &= a + ib \quad \text{where} \quad a, b \in \mathbb{R} \\ \mathbf{z} &= \mathbf{v} + i\mathbf{w} \quad \text{where} \quad \mathbf{v}, \mathbf{w} \in \mathbb{R}^n \end{aligned} \tag{10}$$

Thus, using (9) we have

$$A\mathbf{v} + iA\mathbf{w} = A(\mathbf{v} + i\mathbf{w}) = A\mathbf{z} = \lambda\mathbf{z} = (a + ib)(\mathbf{v} + i\mathbf{w}) = (a\mathbf{v} - b\mathbf{w}) + i(a\mathbf{w} + b\mathbf{v})\,,$$

which, by comparing real and imaginary parts, yields

$$\begin{aligned} A\mathbf{v} &= a\mathbf{v} - b\mathbf{w}\,, \\ A\mathbf{w} &= a\mathbf{w} + b\mathbf{v}\,. \end{aligned} \tag{11}$$

Now, by Lemma 3.4, we have

$$(a\mathbf{v} - b\mathbf{w})\cdot\mathbf{w} = (A\mathbf{v})\cdot\mathbf{w} = \mathbf{v}\cdot(A\mathbf{w}) = \mathbf{v}\cdot(a\mathbf{w} + b\mathbf{v})\,,$$

so

$$a(\mathbf{v}\cdot\mathbf{w}) - b\|\mathbf{w}\|^2 = a(\mathbf{v}\cdot\mathbf{w}) + b\|\mathbf{v}\|^2$$

and hence

$$b(\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2) = 0\,.$$

But $\mathbf{z} = \mathbf{v} + i\mathbf{w}$ is an eigenvector, so the vectors $\mathbf{v}$ and $\mathbf{w}$ cannot both be the zero vector. Therefore $\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 > 0$, and hence

$$b = 0\,.$$

Thus, using (10) and (11), we see that $\lambda = a \in \mathbb{R}$ and $A\mathbf{v} = a\mathbf{v} = \lambda\mathbf{v}$, that is, A has a real eigenvalue with corresponding real eigenvector. □

Lemma 3.6. *Let $\mathbf{x}$ and $\mathbf{y}$ be eigenvectors in $\mathbb{R}^n$ corresponding to two distinct eigenvalues λ and μ of a symmetric matrix $A \in \mathbb{R}^{n\times n}$. Then $\mathbf{x}$ and $\mathbf{y}$ are orthogonal in $\mathbb{R}^n$, that is, $\langle \mathbf{x}, \mathbf{y}\rangle = 0$.*

Proof. We have $\lambda\langle \mathbf{x}, \mathbf{y}\rangle = \langle \lambda\mathbf{x}, \mathbf{y}\rangle = \langle A\mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, A\mathbf{y}\rangle = \langle \mathbf{x}, \mu\mathbf{y}\rangle = \mu\langle \mathbf{x}, \mathbf{y}\rangle$. Since $\lambda \neq \mu$, we must have $\langle \mathbf{x}, \mathbf{y}\rangle = 0$. □

Theorem 3.7 (Spectral Theorem for Symmetric Matrices)**.** *Let $A \in \mathbb{R}^{n\times n}$ be symmetric. Then there is an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ such that*

$$Q^T AQ = D,$$

where $D \in \mathbb{R}^{n\times n}$ is diagonal.

In other words, every real symmetric matrix can be diagonalised by an orthogonal matrix.

Proof. Let $L_A : \mathbb{R}^n \to \mathbb{R}^n$ be the linear map induced by A:

$$L_A(\mathbf{x}) = A\mathbf{x}.$$

Recall that $A = [L_A, \mathcal{E}, \mathcal{E}]$ is just the matrix representing L_A w.r.t. the standard basis $\mathcal{E}$ of $\mathbb{R}^n$. Let $\lambda_1, \ldots, \lambda_k$ be all the distinct eigenvalues of A (as well as L_A). By Lemma 3.6, the corresponding real eigenspaces $E(\lambda_1), \ldots, E(\lambda_k)$ must be mutually orthogonal and in particular, $E(\lambda_i) \cap E(\lambda_j) = \emptyset$.

Form the direct sum

$$H = E(\lambda_1) \oplus \cdots \oplus E(\lambda_k)$$

which is a subspace of $\mathbb{R}^n$. We claim that

$$\mathbb{R}^n = E(\lambda_1) \oplus \cdots \oplus E(\lambda_k).$$

Otherwise, the orthogonal complement $H^\perp \neq \{0\}$. Now we must have $L_A(H^\perp) \subset H^\perp$. Indeed, for each $\mathbf{v} \in H^\perp$ and $h = \mathbf{u}_1 + \cdots + \mathbf{u}_k \in E(\lambda_1) \oplus \cdots \oplus E(\lambda_k) = H$, we have

$$\begin{aligned} \langle L_A(\mathbf{v}), h\rangle &= \langle L_A(\mathbf{v}), \mathbf{u}_1 + \cdots + \mathbf{u}_k\rangle \\ &= \langle \mathbf{v}, L_A(\mathbf{u}_1 + \cdots + \mathbf{u}_k)\rangle \\ &= \langle \mathbf{v}, \lambda_1\mathbf{u}_1 + \cdots + \lambda_k\mathbf{u}_k\rangle \\ &= \lambda_1\langle \mathbf{v}, \mathbf{u}_1\rangle + \cdots + \lambda_k\langle \mathbf{v}, \mathbf{u}_k\rangle = 0 \end{aligned}$$

which gives $L_A(\mathbf{v}) \in H^\perp$. Hence we can restrict L_A to a linear map on $H^\perp$, denoted by $L_A|_{H^\perp} : H^\perp \to H^\perp$. By Lemma 3.5, $L_A|_{H^\perp}$ has an eigenvector $\mathbf{z}$ in $H^\perp \neq \{0\}$ with eigenvalue μ. Moreover, $\mu \neq \lambda_1, \ldots, \lambda_k$ because $\mathbf{z}$ is not in the eigenspaces $E(\lambda_1), \ldots, E(\lambda_k)$. This is impossible since $\lambda_1, \ldots, \lambda_k$ are all the distinct eigenvalues of A.

Now $\mathbb{R}^n = E(\lambda_1) \oplus \cdots \oplus E(\lambda_k)$ implies that we can find an orthonormal basis $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of $\mathbb{R}^n$ which is the union of orthonormal bases of the eigenspaces $E(\lambda_1), \ldots, E(\lambda_k)$.

The matrix $D = [L_A, B, B]$ representing L_A w.r.t. the basis B must be diagonal with diagonal entries $\{\lambda_1, \ldots, \lambda_k\}$ (cf. Coursework 9, Exercese 4). By the change-of-basis theorem, we have

$$D = [L_A, B, B] = M^B_{\mathcal{E}}[L_A, \mathcal{E}, \mathcal{E}]M^{\mathcal{E}}_B = M^B_{\mathcal{E}} AM^{\mathcal{E}}_B$$

where the transition matrix $Q = M_B^{\mathcal{E}}$ is invertible and is given by

$$Q = ([\mathbf{v}_1]_{\mathcal{E}} \cdots [\mathbf{v}_n]_{\mathcal{E}}).$$

It follows that the (i, j)-entry of $Q^T Q$ is given by

$$\begin{aligned}(Q^T Q)_{ij} &= (i\text{-th row of } Q^T)(j\text{-th column of } Q) = (i\text{-th column of } Q)(j\text{-th column of } Q) \\ &= \langle \mathbf{v}_i, \mathbf{v}_j \rangle = \delta_{ij}\end{aligned}$$

since $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is orthonormal. This proves that $Q^T Q = I$, that is, Q is orthogonal and also $Q^{-1} A Q = D$. □

Let us record the following consequence of the Spectral Theorem, which is of independent interest:

Corollary 3.8. *The eigenvalues of a symmetric matrix A are real, and eigenvectors corresponding to distinct eigenvalues are orthogonal.*

Proof. Combine the Diagonalisation Theorem and the Spectral Theorem. □

Example 3.9. Consider the symmetric matrix

$$A = \begin{pmatrix} 0 & 2 & -1 \\ 2 & 3 & -2 \\ -1 & -2 & 0 \end{pmatrix}.$$

Find an orthogonal matrix Q that diagonalises A.

Solution. The characteristic polynomial of A is

$$\det(A - \lambda I) = -\lambda^3 + 3\lambda^2 + 9\lambda + 5 = (1 + \lambda)^2 (5 - \lambda),$$

so the eigenvalues of A are -1 and 5. Computing $N(A+I)$ in the usual way shows that $\{\mathbf{x}_1, \mathbf{x}_1\}$ is a basis for $N(A + I)$ where

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.$$

Similarly, we find that the eigenspace $N(A - 5I)$ corresponding to the eigenvalue 5 is 1-dimensional with basis

$$\mathbf{x}_3 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}.$$

In order to construct the diagonalising orthogonal matrix for A it suffices to find orthonormal bases for each of the eigenspaces, since eigenvectors corresponding to distinct eigenvalues are orthogonal.

To find an orthonormal basis for $N(A+I)$ we apply the Gram Schmidt process to the basis $\{\mathbf{x}_1, \mathbf{x}_2\}$ to produce the orthogonal set $\{\mathbf{v}_1, \mathbf{v}_2\}$:

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

Now $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{x}_3\}$ is an orthogonal basis of $\mathbb{R}^3$ consisting of eigenvectors of A, so normalising them to produce

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}, \quad \mathbf{u}_2 = \frac{1}{\|\mathbf{v}_2\|}\mathbf{v}_2 = \begin{pmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}, \quad \mathbf{u}_3 = \frac{1}{\|\mathbf{x}_3\|}\mathbf{x}_3 = \begin{pmatrix} -1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix},$$

allows us to write down the orthogonal matrix

$$Q = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{pmatrix}$$

which diagonalises A, that is,

$$Q^T A Q = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

□