0.1 (Linear) span of vectors

Definition 0.1.1. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be vectors in a vector space V. A *linear combination* of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a vector $\mathbf{v}$ of the form

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n$$

where $\alpha_1, \ldots, \alpha_n$ are scalars. The set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is called the *span* of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and is denoted by $\operatorname{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$, that is,

$$\operatorname{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \{\, \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n \mid \alpha_1, \ldots, \alpha_n \in \mathbb{R} \,\} \ .$$

Example 0.1.2. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \in \mathbb{R}^3$ be given by

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} .$$

Determine $\operatorname{Span}(\mathbf{e}_1, \mathbf{e}_2)$ and $\operatorname{Span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$.

Solution. Since

$$\alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{pmatrix} , \quad \text{while} \quad \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} ,$$

we see that

$$\operatorname{Span}(\mathbf{e}_1, \mathbf{e}_2) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, x_3 = 0 \right\} , \quad \text{while} \quad \operatorname{Span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \mathbb{R}^3 .$$

□

Theorem 0.1.3. *Let* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *be vectors in a vector space* V*. Then* $\operatorname{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ *is a subspace of* V*.*

Definition 0.1.4. Let V be a vector space, and let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$. We say that the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a *spanning set* for V if

$$\operatorname{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n) = V \ .$$

If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a spanning set for V, we shall also say that $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ **spans** V, that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ **span** V or that V is **spanned** by $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

Notice that the above definition can be rephrased as follows. A set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a spanning set for V, if and only if every vector in V can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

Example 0.1.5. Show that $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is a spanning set for P_2, where

$$\mathbf{p}_1(x) = 2 + 3x + x^2\,, \quad \mathbf{p}_2(x) = 4 - x\,, \quad \mathbf{p}_3(x) = -1\,.$$

Solution. Let $\mathbf{p}$ be an arbitrary polynomial in P_2, say, $\mathbf{p}(x) = a + bx + cx^2$. We need to show that it is possible to find weights α_1, α_2 and α_3 such that

$$\alpha_1\mathbf{p}_1 + \alpha_2\mathbf{p}_2 + \alpha_3\mathbf{p}_3 = \mathbf{p}\,,$$

that is

$$\alpha_1(2 + 3x + x^2) + \alpha_2(4 - x) - \alpha_3 = a + bx + cx^2\,.$$

Comparing coefficients we find that the weights have to satisfy the system

$$\begin{array}{rcrcrcl} 2\alpha_1 & + & 4\alpha_2 & - & \alpha_3 & = & a \\ 3\alpha_1 & - & \alpha_2 & & & = & b \\ \alpha_1 & & & & & = & c \end{array}$$

The coefficient matrix is nonsingular, so the system must have a unique solution for all choices of a, b, c. In fact, using back substitution yields $\alpha_1 = c$, $\alpha_2 = 3c - b$, $\alpha_3 = 14c - 4b - a$. Thus $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is a spanning set for P_2. $\square$

Example 0.1.6. Find a spanning set for $N(A)$, where

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}.$$

Proof. We have already calculated $N(A)$ for this matrix in Example **??**, and found that

$$N(A) = \left\{ \alpha \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \,\middle|\, \alpha, \beta, \gamma \in \mathbb{R} \right\}.$$

Thus, $\left\{(2,1,0,0,0)^T, (1,0,-2,1,0)^T, (-3,0,2,0,1)^T\right\}$ is a spanning set for $N(A)$. $\square$

0.2 Linear independence

Definition 0.2.1. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be vectors in a vector space V. They are said to be **linearly dependent** if there exist scalars $c_1, \ldots, c_n$, not all zero, such that

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}\,.$$

Definition 0.2.2. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of vectors in a vector space V is said to be **linearly independent** if they are not linearly dependent, that is, if

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0} \;\;\Rightarrow\;\; c_1, \ldots, c_n = 0.$$

Example 0.2.3. The vectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathbb{R}^2$ are linearly independent. In order to see this, suppose that

$$c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\,.$$

Then c_1 and c_2 must satisfy the 2×2 system

$$\begin{array}{ccccc} 2c_1 & + & c_2 & = & 0 \\ c_1 & + & c_2 & = & 0 \end{array}$$

However, as is easily seen, the only solution of this system is $c_1 = c_2 = 0$. Thus, the two vectors are indeed linearly independent as claimed.

Example 0.2.4. Let $\mathbf{p}_1, \mathbf{p}_2 \in P_1$ be given by

$$\mathbf{p}_1(t) = 2 + t\,, \quad \mathbf{p}_2(t) = 1 + t\,.$$

Then $\mathbf{p}_1$ and $\mathbf{p}_2$ are linearly independent. In order to see this, suppose that

$$c_1\mathbf{p}_1 + c_2\mathbf{p}_2 = \mathbf{0}\,.$$

Then, for all t

$$c_1(2+t) + c_2(1+t) = 0\,,$$

so, for all t

$$(2c_1 + c_2) + (c_1 + c_2)t = 0\,.$$

Notice that the polynomial on the left-hand side of the above equation will be the zero polynomial if and only if its coefficients vanish, so c_1 and c_2 must satisfy the 2×2 system

$$\begin{array}{rcrcl} 2c_1 & + & c_2 & = & 0 \\ c_1 & + & c_2 & = & 0 \end{array}$$

However, as in the previous example, the only solution of this system is $c_1 = c_2 = 0$. Thus $\mathbf{p}_1$ and $\mathbf{p}_2$ are indeed linearly independent as claimed.

Example 0.2.5 (Geometric interpretation of linear independence in $\mathbb{R}^2$ and $\mathbb{R}^3$)**.**

(a) If $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent in $\mathbb{R}^2$ then

$$c_1\mathbf{x} + c_2\mathbf{y} = \mathbf{0}\,,$$

where c_1 and c_2 are not both 0. If, say $c_1 \neq 0$, then

$$\mathbf{x} = -\frac{c_2}{c_1}\mathbf{y}\,.$$

Thus one of the vectors must be a scalar multiple of the other, or, put differently, the two vectors must be collinear.

Conversely, if two vectors in $\mathbb{R}^2$ are not collinear, they are linearly independent.

(b) Just as in $\mathbb{R}^2$, two vectors in $\mathbb{R}^3$ are linearly dependent if and only if they are collinear. Suppose now that $\mathbf{x}$ and $\mathbf{y}$ are two linearly independent vectors in $\mathbb{R}^3$. Since they are not collinear, they will span a plane (through the origin). If $\mathbf{z}$ is another vector lying in this plane, then $\mathbf{0}$ can be written as a linear combination of $\mathbf{x}$ and $\mathbf{y}$, hence $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are linearly dependent. Conversely, if $\mathbf{z}$ does not lie in the plane spanned by $\mathbf{x}$ and $\mathbf{y}$, then $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are linearly independent.

In other words, three vectors in $\mathbb{R}^3$ are linearly independent if and only if they are not coplanar.

Theorem 0.2.6. *Let $\mathbf{x}_1, \dots, \mathbf{x}_n$ be n vectors in $\mathbb{R}^n$ and let $A \in \mathbb{R}^{n \times n}$ be the matrix whose j-th column is $\mathbf{x}_j$. Then the vectors $\mathbf{x}_1, \dots, \mathbf{x}_n$ are linearly dependent if and only if A is singular.*

Proof. The equation

$$c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n = \mathbf{0}$$

can be written as

$$A\mathbf{c} = \mathbf{0}\,, \quad \text{where} \quad \mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

This system has a non-trivial solution $\mathbf{c} \neq \mathbf{0}$ if and only A is singular. □

Corollary 0.2.7. *In the above theorem, the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent if and only if A is invertible, which is equivalent to the fact that A can be reduced to row echelon form with exactly n leading columns.*

Example 0.2.8. Determine whether the following three vectors in $\mathbb{R}^3$ are linearly independent:

$$\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}.$$

Solution. Since

$$\begin{vmatrix} -1 & 5 & 4 \\ 3 & 2 & 5 \\ 1 & 5 & 6 \end{vmatrix} = \begin{vmatrix} -1 & 3 & 1 \\ 5 & 2 & 5 \\ 4 & 5 & 6 \end{vmatrix} \overset{R_1+R_2}{=} \begin{vmatrix} 4 & 5 & 6 \\ 5 & 2 & 5 \\ 4 & 5 & 6 \end{vmatrix} \overset{R_1-R_3}{=} \begin{vmatrix} 0 & 0 & 0 \\ 5 & 2 & 5 \\ 4 & 5 & 6 \end{vmatrix} = 0\,,$$

the vectors are linearly dependent. □

Theorem 0.2.9. *Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be vectors in a vector space V. A vector $\mathbf{v} \in \operatorname{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ can be written uniquely as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ if and only if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.*

Proof. If $\mathbf{v} \in \operatorname{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ then $\mathbf{v}$ can be written

$$\mathbf{v} = \alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n\,, \tag{1}$$

for some scalars $\alpha_1, \ldots, \alpha_n$. Suppose that $\mathbf{v}$ can also be written in the form

$$\mathbf{v} = \beta_1\mathbf{v}_1 + \cdots + \beta_n\mathbf{v}_n\,, \tag{2}$$

for some scalars $\beta_1, \ldots, \beta_n$. We start by showing that if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent, then $\alpha_i = \beta_i$ for every $i = 1, \ldots, n$ (that is, the representation

(1) is unique). To see this, suppose that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent. Then subtracting (2) from (1) gives

$$(\alpha_1 - \beta_1)\mathbf{v}_1 + \cdots + (\alpha_n - \beta_n)\mathbf{v}_n = \mathbf{0}\,, \tag{3}$$

which forces $\alpha_i = \beta_i$ for every $i = 1, \ldots, n$ as desired.

Conversely, if the representation (1) is not unique, then there must be a representation of the form (2) where $\alpha_i \neq \beta_i$ for some i between 1 and n. But then (3) means that there exists a non-trivial linear dependence between $\mathbf{v}_1, \ldots, \mathbf{v}_n$, so these vectors are linearly dependent. □

0.3 Basis and dimension

The concept of a basis and the related notion of dimension are among the key ideas in the theory vector of spaces, of immense practical and theoretical importance.

Definition 0.3.1. A set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of vectors forms a *basis* for a vector space V if

(i) $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent;

(ii) $\operatorname{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n) = V$.

In other words, a basis for a vector space is a 'minimal' spanning set, in the sense that it contains no superfluous vectors: every vector in V can be written as a linear combination of the basis vectors (because of property (ii)), and there is no redundancy in the sense that no basis vector can be expressed as a linear combination of the other basis vectors (by property (i)).

Example 0.3.2. Let

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis for $\mathbb{R}^3$, called the *standard basis*.

Indeed, as is easily seen, every vector in $\mathbb{R}^3$ can be written as a linear combination of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and, moreover, the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are linearly independent.

Example 0.3.3.
$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$
is a basis for $\mathbb{R}^3$.

First, note that the vectors are linearly independent since the matrix
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$
is in echelon form with 3 leading columns. Moreover, the vectors span $\mathbb{R}^3$ since, if $(a, b, c)^T$ is an arbitrary vector in $\mathbb{R}^3$, then
$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (a-b) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (b-c) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} .$$

The previous two examples show that a vector space may have more than one basis.

Example 0.3.4. Let
$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} .$$

Then $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ is a basis for $\mathbb{R}^{2\times 2}$, because the four vectors span $\mathbb{R}^{2\times 2}$ (as was shown in Coursework 5, Exercise 7(b)) and they are linearly independent. To see this, suppose that
$$c_1 E_{11} + c_2 E_{12} + c_3 E_{21} + c_4 E_{22} = O_{2\times 2} .$$
Then
$$\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$
so $c_1 = c_2 = c_3 = c_4 = 0$.

Most of the vector spaces we have encountered so far have particularly simple bases, termed ‘standard bases’:

Example 0.3.5 (Standard bases for $\mathbb{R}^n$, $\mathbb{R}^{m\times n}$ and P_n).

$\mathbb{R}^n$: The n columns of I_n form the standard basis of $\mathbb{R}^n$, usually denoted by $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$.

$\mathbb{R}^{m\times n}$: A canonical basis can be constructed as follows. For $i = 1, \ldots, m$ and $j = 1, \ldots, n$ let $E_{ij} \in \mathbb{R}^{m\times n}$ be the matrix whose (i, j)-entry is 1, and all other entries are 0. Then $\{\, E_{ij} \mid i = 1, \ldots, m\,, j = 1, \ldots, n\,\}$ is the standard basis for $\mathbb{R}^{m\times n}$.

P_n: The standard basis is the collection $\{\mathbf{p}_0, \ldots, \mathbf{p}_n\}$ of all monomials of degree less than n, that is,

$$\mathbf{p}_k(t) = t^k\,, \qquad \text{for } k = 0, \ldots, n.$$

If this is not clear to you, you should check that it really is a basis!

Theorem 0.3.6. *Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be vectors in a vector space V. If* $\operatorname{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n) = V$*, then any collection of m vectors in V where $m > n$ is linearly dependent.*

Proof. Let $\mathbf{u}_1, \ldots, \mathbf{u}_m$ be m vectors in V where $m > n$. Then, since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ span V, we can write

$$\mathbf{u}_i = \alpha_{i1}\mathbf{v}_1 + \cdots + \alpha_{in}\mathbf{v}_n \qquad \text{for } i = 1, \ldots, m\,.$$

Thus, a linear combination of the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_m$ can be written as

$$\begin{aligned} c_1\mathbf{u}_1 + \cdots + c_m\mathbf{u}_m &= \sum_{i=1}^m c_i\mathbf{u}_i \\ &= \sum_{i=1}^m c_i \left(\sum_{j=1}^n \alpha_{ij}\mathbf{v}_j\right) \\ &= \sum_{j=1}^n \left(\sum_{i=1}^m \alpha_{ij}c_i\right)\mathbf{v}_j\,. \end{aligned}$$

Now consider the system of n equations for the m unknowns $c_1, \ldots, c_m$

$$\sum_{i=1}^m \alpha_{ij}c_i = 0 \quad \text{for } j = 1, \ldots, n.$$

This is a homogeneous system with more unknowns than equations, so by Theorem **??** it must have a non-trivial solution $(\hat{c}_1, \ldots, \hat{c}_m)^T$. But then

$$\hat{c}_1\mathbf{u}_1 + \cdots + \hat{c}_m\mathbf{u}_m = \sum_{j=1}^n 0\mathbf{v}_j = \mathbf{0}\,,$$

so $\mathbf{u}_1, \ldots, \mathbf{u}_m$ are linearly dependent. □

Corollary 0.3.7. *If a vector space V has a basis of n vectors, then every basis of V must have exactly n vectors.*

Proof. Suppose that $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ are both bases for V. We shall show that $m = n$. In order to see this, notice that, since $\operatorname{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n) = V$ and $\mathbf{u}_1, \ldots, \mathbf{u}_m$ are linearly independent it follows by the previous theorem that $m \leq n$. By the same reasoning, since $\operatorname{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_m) = V$ and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent, we must have $n \leq m$. So, all in all, we have $n = m$, that is, the two bases have the same number of elements. □

In view of this corollary it now makes sense to talk about *the* number of elements of a basis, and give it a special name:

Definition 0.3.8. Let V be a vector space. If V has a basis consisting of n vectors, we say that V has *dimension* n, and write $\dim V = n$.

The vector space $\{\mathbf{0}\}$ is said to have dimension 0. The vector space V is said to be **finite dimensional** if there is a finite set of vectors spanning V; otherwise it is said to be **infinite dimensional** .

Example 0.3.9. By Example 0.3.5 the vector spaces $\mathbb{R}^n$, $\mathbb{R}^{m\times n}$ and P_n are finite dimensional with dimensions

$$\dim \mathbb{R}^n = n\,, \quad \dim \mathbb{R}^{m\times n} = mn\,, \quad \dim P_n = n+1\,.$$

As an example of an infinite dimensional vector space, consider the vector space P of all polynomials with real coefficients. Note that any finite collection of monomials is linearly independent, so P must be infinite dimensional. For the same reason, $C[a,b]$ and $C^1[a,b]$ are infinite dimensional vector spaces.

Theorem 0.3.10. *If V is a vector space with $\dim V = n$, then:*

(a) any set consisting of n linearly independent vectors spans V;

(b) any n vectors that span V are linearly independent.

Remark 0.3.11. The above theorem provides a convenient tool to check whether a set of vectors forms a basis. The theorem tells us that n linearly independent vectors in an n-dimensional vector space are automatically spanning, so these vectors are a basis for the vector space. This is often useful in situations where linear independence is easier to check than the spanning property.

0.4 Row space and column space

Definition 0.4.1. Let $A \in \mathbb{R}^{m\times n}$.

- The subspace of $\mathbb{R}^{1\times n}$ spanned by the row vectors of A is called the *row space* of A and is denoted by $\mathrm{row}(A)$.
- The subspace of $\mathbb{R}^{m\times 1}$ spanned by the column vectors of A is called the *column space* of A and is denoted by $\mathrm{col}(A)$.

Example 0.4.2. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

- Since
$$\alpha \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \alpha & \beta & 0 \end{pmatrix}$$
$\mathrm{row}(A)$ is a 2-dimensional subspace of $\mathbb{R}^{1\times 3}$.
- Since
$$\alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$
$\mathrm{col}(A)$ is a 2-dimensional subspace of $\mathbb{R}^{2\times 1}$.

Notice that the row space and column space of a matrix are generally distinct objects. Indeed, one is a subspace of $\mathbb{R}^{1\times n}$ the other a subspace of $\mathbb{R}^{m\times 1}$. However, in the example above, both spaces have the same dimension (namely 2). In fact, this is always the case.

Theorem 0.4.3. *Let $A \in \mathbb{R}^{m\times n}$. Then*

$$\dim \mathrm{row}(A) = \dim \mathrm{col}(A) .$$

Definition 0.4.4. The *rank* of a matrix, denoted by $\mathrm{rank}\, A$, is the dimension of the row space (which is the same as the dimension of the column space).

How does one calculate the rank of a matrix? The next result provides the clue:

Theorem 0.4.5. *Let $A \in \mathbb{R}^{m\times n}$. Then A is row equivalent to a matrix U in echelon form, and the nonzero rows of U form a basis for row(A).*

Proof. Apply a sequence of elementary row operations on A to obtain a matrix U in echelon form. So A is row equivalent to U with $row(A) = row(U)$.

Let $R_1, \dots, R_k$ be nonzero rows of U which clearly span $row(U)$. They form a basis if they are linearly independent. Suppose not, then there exist scalars

$$(\alpha_1, \dots, \alpha_k) \neq (0, \dots, 0)$$

such that

$$\alpha_1 R_1 + \cdots + \alpha_k R_k = \mathbf{0}.$$

Let α_j be the first nonzero scalar. Then

$$\alpha_j R_j + \alpha_{j+1} R_{j+1} + \cdots + \alpha_k R_k = \mathbf{0}.$$

But $R_{j+1}, R_{j+2} \dots, R_k$ all start with more zeros than R_j which implies $\alpha_j = 0$, giving a contradiction.

Hence $R_1, \dots, R_k$ are linearly independent and form a basis. $\square$

To find a basis for the row space and the rank of a matrix A:

- bring matrix to row echelon form U;
- the nonzero rows of U will form a basis for $\mathrm{row}(A)$;
- the number of nonzero rows of U equals $\operatorname{rank} A$.

Example 0.4.6. Let

$$A = \begin{pmatrix} 1 & -3 & 2 \\ 1 & -2 & 1 \\ 2 & -5 & 3 \end{pmatrix}.$$

Then

$$\begin{pmatrix} 1 & -3 & 2 \\ 1 & -2 & 1 \\ 2 & -5 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus

$$\left\{ \begin{pmatrix} 1 & -3 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \end{pmatrix} \right\}$$

is a basis for $\mathrm{row}(A)$, and $\operatorname{rank} A = 2$.

To find a basis for the column space of a matrix A:

- bring A to row echelon form and identify the leading variables;
- the columns of A containing the leading variables form a basis for $\operatorname{col}(A)$.

Example 0.4.7. Let

$$A = \begin{pmatrix} 1 & -1 & 3 & 2 & 1 \\ 1 & 0 & 1 & 4 & 1 \\ 2 & -1 & 4 & 7 & 4 \end{pmatrix} .$$

Then the row echelon form of A is

$$\begin{pmatrix} 1 & -1 & 3 & 2 & 1 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} .$$

The leading variables are in columns 1,2, and 4. Thus a basis for $\operatorname{col}(A)$ is given by

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix} \right\} .$$

It turns out that the rank of a matrix A is intimately connected with the dimension of its nullspace $N(A)$. Before formulating this relation, we require some more terminology:

Definition 0.4.8. If $A \in \mathbb{R}^{m\times n}$, then $\dim N(A)$ is called the **nullity** of A, and is denoted by $\operatorname{nul} A$.

Example 0.4.9. Find the nullity of the matrix

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix} .$$

Solution. Reduce A to row echelon form U and then using back substitution to solve $U\mathbf{x} = \mathbf{0}$, giving

$$N(A) = \{\, \alpha\mathbf{x}_1 + \beta\mathbf{x}_2 + \gamma\mathbf{x}_3 \mid \alpha, \beta, \gamma \in \mathbb{R} \,\} ,$$

where

$$\mathbf{x}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}.$$

It is not difficult to see that $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly independent, so $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a basis for $N(A)$. Thus, $\operatorname{nul} A = 3$. □

Note that in the above example the nullity of A is equal to the number of free variables of the system $A\mathbf{x} = \mathbf{0}$. This is no coincidence, but true always!

The connection between the rank and nullity of a matrix, alluded to above, is the content of the following beautiful theorem.

Theorem 0.4.10 (Rank-Nullity Theorem)**.** *If $A \in \mathbb{R}^{m \times n}$, then*

$$\operatorname{rank} A + \operatorname{nul} A = n\,.$$

Proof. Bring A to row echelon form U. Write $r = \operatorname{rank} A$. Now observe that U has r non-zero rows, hence $U\mathbf{x} = \mathbf{0}$ has $n - r$ free variables, so $\operatorname{nul} A = n - r$. □