MATH 3305 General Relativity Problem sheet 3
Please hand in your solutions Friday, 30 October 2009

Problem 1 (20 points) Show that the line element

$$ds^2 = g_{ij} dX^i dX^j \tag{1}$$

of a metric tensor does not depend on local coordinates $(X^1, \ldots, X^n)$.

Problem 2 (30 points) In the lectures we used the total derivative to transform the Euclidean line element

$$ds^2 = dx^2 + dy^2$$

in Cartesian coordinates into the line element

$$ds^2 = dr^2 + r^2 d\theta^2$$

in polar coordinates. Obtain the same result by starting from the transformation rule for a $\binom{0}{2}$-tensor

$$\widetilde{g}_{ij} = \frac{\partial X^r}{\partial \widetilde{X}^i} \frac{\partial X^s}{\partial \widetilde{X}^j} g_{rs}.$$

Which derivation do you think is easier?

Problem 3 (30 points) Let us consider the plane $\mathbb{R}^2$ with the Euclidean metric. Derive a formula for the length of a curve $\gamma\colon [a,b] \to \mathbb{R}^2$ given in polar coordinates $\gamma(t) = (r(t), \theta(t))$ by

(a) the theory from this course,

(b) starting from the usual definition of length of a curve in $\mathbb{R}^2$.

Problem 4 (20 points) Simplify expressions

$$\delta^i_i, \quad g^{ji} g_{ij}, \quad A_{ij} V^i V^j, \quad g^{ij} A_{ij}.$$

where g_{ij} is a Riemann metric, V^i is a vector, and A_{ij} is a $\binom{0}{2}$-tensor that is *anti-symmetric*, that is, $A_{ij} = -A_{ji}$. (Hint: All four expressions have been written using the Einstein summing convention.)