MATH 3305 General Relativity Problem sheet 1

Please hand in your solutions by Friday, 15 October 2009.

Problem 1 (30 points) Let $\mathcal{M}$ be a manifold. Let V^a be contravariant vector and let W_a be a covariant vector. Show that

$$\mu = V^a W_a \tag{1}$$

is a scalar. (Hint: How does μ transform under coordinate transformations?)

Problem 2 (20 points) Determine which of the following tensor equations are valid, and describe possible errors

$$K = R_{abcd} R^{abcd} \tag{2}$$

$$T_{ab} = F_{ac} F^c{}_c + \frac{1}{4}\eta_{ab} F_{ab} F^{ab} \tag{3}$$

$$R_{ab} - \frac{1}{2}R = 8\pi\kappa T_{ab} \tag{4}$$

$$E_a{}^b = F_{ac} H^{cb}. \tag{5}$$

Problem 3 (30 points) Let $\mathcal{M}$ be a 4-dimensional manifold and T_{ab} be type $(0,2)$ tensor. How many independent components does T_{ab} have in general? We defined

$$T_{(ab)} = \frac{1}{2}\left(T_{ab} + T_{ba}\right) \tag{6}$$

$$T_{[ab]} = \frac{1}{2}\left(T_{ab} - T_{ba}\right). \tag{7}$$

How many independent components do $T_{(ab)}$ and $T_{[ab]}$ have, respectively? Confirm that their sum adds up to the total number of independent components of T_{ab}.

Problem 4 (20 points) (Recall classical mechanics). Let $L(x(t), \dot{x}(t))$ be a smooth function of $x(t)$ and $\dot{x}(t) = dx(t)/dt$. What differential equation must L satisfy to extremises the following functional

$$S = \int L(x, \dot{x}) dt\,? \tag{8}$$

(Keywords: Hamilton's/action principle, Euler-Lagrange equations, variational calculus)