LECTURE 24

24.1. The Einstein field equations (EFEs)

It seems to be a good idea to relate the curvature Riemann tensor tensor with the stress-energy tensor? Unfortunately the rang of this tensor is 4, which is too big in comparison with rang 2 for the stress-energy tensor. To solve this problem we can construct the tensor of the second rank like this:

$$R_{ik} = g^{lm} R_{limk} = R^{l}_{ilk}$$

This tensor is called the Ricci tensor. Then we even can construct a zero rank tensor, i.e. a scalar

$$R = g^{ik} R_{ik},$$

which is called the scalar curvature.
Einstein (with help of Gilbert) introduced the following tensor

$$G_{ik} = R_{ik} - \frac{1}{2} g_{ik} R,$$

which is called the Einstein'tensor.
Now the Einstein Field Equations (EFE) can be written as

$$G_{ik} = \kappa T_{ik},$$

where T_{ik} is the stress-energy tensor (sometimes stress-energy-momentum tensor), which describes the density and flux of energy and momentum.

24.2. The Stress-Energy Tensor.

In general Relativity, as well as in all other parts of Physics, the stress-energy tensor is symmetric and contains ten independent components:

Component	What it represents	Number of components
T_{00}	the energy density	1
$T_{0\alpha}$	the flux of energy	3
$T_{\alpha 0} \equiv T_{0\alpha}$	the density of the momentum	
$T_{\alpha\alpha}$	the pressure (if it positive) or tension (if it negative)	3
$T_{\alpha\beta}, \alpha \neq \beta$	shear stress	3

All these ten components participate in generation of gravitational field, while in Newton gravity the only source of gravitational field is the mass density.

24.3. The Equation of State in Cosmology

In cosmology we study the nearly isotropic Universe, in this case all out of diagonal components are equal to zero and we have

$$T_{ik} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix},$$

where ϵ is the energy density and P is a pressure or tension. In cosmology these two quantities are related be the equation called The equation of State:

$$p = \alpha\epsilon,$$

where α is a constant called the equation of state parameter.

α	Substance
0	Dust
$1/3$	CMB
$-1/3$	Cosmic strings (not examinable)
$-1 < \alpha < -1/3$	Dark energy
-1	Dark energy in form of the Λ-term
$\alpha < -1$	Fantom energy (not examinable)

24.4. Self-consistency of the EFEs

The constant κ in EFE is called the Einstein constant. To determine this constant we can use so called the correspondence principle, which states that the EFEs in weak-field and slow-motion approximation should reduce to Newton's law of gravity, described by the Poisson's equation)

$$\Delta\phi = 4\pi G\rho$$

According to this obvious principle

$$\kappa = \frac{8\pi G}{c^4}.$$

Thus the EFEs can be written as

$$\mathbf{R_{ik}} - \frac{\mathbf{1}}{\mathbf{2}}\mathbf{g_{ik}}\mathbf{R} = \frac{\mathbf{8\pi G}}{\mathbf{c^4}}\mathbf{T_{ik}}.$$

Despite the simple appearance of the equations they are, in fact, quite complicated. The EFEs are 10 equations for 10 independent components of the metric tensor g_{ik}. Taking into account that the Ricci, R_{ik}, as well as the scalar curvature, R, contain linear combinations of second partial derivatives of the metric tensor and nonlinear combinations of its first derivatives, we can see thatthe EFEs is a system of 10 coupled, nonlinear, hyperbolic-elliptic partial differential equations.

As follows from the Bianki identiti

$$R^l_{m;\,l} = \frac{1}{2}R,m,$$

hence

$$T^i_{k;i} = 0,$$

which means that according to the EFEs the covariant divergence of the stress-energy tensor is equal to zero, i.e. the EFEs contain all conservation laws which from mathematical point of view are equivalent to all dynamical equations for matter and fields. In order words, the EFEs describe in self-consistent way the distribution and motion of matter and fields in curved space-time, while the geometry of the space-time itself is determined trough the EFEs by the distribution and the motion of matter and fields.