LECTURE 23

23.1. Properties of the Riemann curvature tensor

With the help of the Riemann tensor we can differentiate twice tensors of arbitrary rank, for example,

$$A_{i;\,k;\,l} - A_{i;\,l;\,k} = A_m R^m_{ikl}$$

$$A_{ik;\,l;\,m} - A_{ik;\,m;\,l} = A_{in} R^n_{klm} + A_{nk} R^n_{ilm}.$$

The Riemann curvature tensor appears in so called the geodesic deviation equation. This equation measures the change in separation of neighbouring geodesics. In the language of mechanics it measures the rate of relative acceleration of two particles moving forward on neighbouring geodesics, separated by a 4-vector η^i:

$$\frac{d^2\eta^i}{ds^2} = R^i_{klm} u^k u^l \eta^m,$$

where

$$u^i = \frac{dx^i}{ds},$$

If gravitational field is weak and all motions are slow

$$u^i \approx \delta^i_0,$$

the above equation is reduced to the Newtonian equation for the tidal acceleration.

A G Polnarev. Mathematical aspects of cosmology (MAS347), 2008. III. Mathematical structure of General Relativity, Lecture 23. 23.1. Properties of The Riemann curvature tensor

Let us introduce covariant version of the Riemann tensor

$$R_{iklm} = g_{in} R^n_{klm}.$$

One can easily show that

$$R_{iklm} = \frac{1}{2}\left(g_{im,k,l} + g_{kl,i,m} - g_{il,k,m} - g_{km,i,l}\right) + g_{np}\left(\Gamma^n_{kl}\Gamma^p_{im} - \Gamma^n_{km}\Gamma^p_{il}\right).$$

The following properties of the curvature tensor R_{iklm} are important for derivation of EFE:

$$\Rightarrow 1)\;\; R_{iklm} = -R_{kilm} = -R_{ikml},$$
$$2)\;\; R_{iklm} + R_{imkl} + R_{ilmk} = 0,$$
$$3)\;\; R^n_{ikl;\,m} + R^n_{imk;\,l} + R^n_{ilm;\,k} = 0.$$

The property 3) is called Bianchy identity and can be proofed in locally-geodesic coordinate system, where

$$\Gamma^i_{kl} = 0$$

and

$$R^n_{ikl;m} = R^n_{ikl,m} = \Gamma^n_{il,m,k} - \Gamma^n_{ik,m,l}.$$

23.2. The Ricci tensor

Replacing k by l and l by k and then just putting $m = l$ we can obtain a tensor of second rank, called the Ricci tensor:

$$R_{ik} = g^{lm} R_{limk} = R^l_{ilk}$$
$$R_{ik} = \Gamma^l_{ik,l} - \Gamma^l_{il,k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km}.$$

$$R_{ik} = R_{ki}$$
$$R = g^{ik} R_{ik} = g^{il} g^{km} R_{iklm}$$