LECTURE 22

The Riemann curvature tensor

We know that

$$A_{i,k,l} - A_{i,l,k} = 0.$$

What can we say about the following commutator

$$A_{i;\,k;\,l} - A_{i;\,l;\,k}?$$

Straightforward calculations show that this is not equal to zero in the presence of gravitational field and can be presented as

$$A_{i;\,k;\,l} - A_{i;\,l;\,k} = A_m R^m_{ikl},$$

where

$$R^i_{klm} = \Gamma^i_{km,l} - \Gamma^i_{kl,m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl}.$$

The object R^i_{klm} obviously is a tensor and called the curvature Riemann tensor. We know that if at least one component of a tensor is not equal to zero at least in one frame of reference, the same is true for any other frame of reference, in other words, tensors can not be eliminated by transformations of coordinates.
The Riemann tensor describes actual tidal gravitational field, which is not local and, hence, can not be eliminated even in the locally inertial frame of reference. Let us calculate the curvature Riemann tensor directly:

$$
\begin{gathered}
A_{i;k;l} - A_{i;l;k} = \\
A_{i;k,l} - \Gamma^m_{li} A_{m;k} - \Gamma^m_{lk} A_{i;m} - \\
- A_{i;l,k} + \Gamma^m_{ki} A_{m;l} + \Gamma^m_{kl} A_{i;m} = \\
\left(A_{i,k} - \Gamma^m_{ik} A_m\right)_{,l} - \Gamma^m_{li}\left(A_{m,k} - \Gamma^n_{mk} A_n\right) - \\
- \left(A_{i,l} - \Gamma^m_{il} A_m\right)_{,k} + \Gamma^m_{ki}\left(A_{m,l} - \Gamma^n_{ml} A_n\right) = \\
A_{i,k,l} - A_{i,l,k} - \Gamma^m_{ik} A_{m,l} - \Gamma^m_{il} A_{m,k} - \Gamma^m_{kl} A_{i,m} + \Gamma^m_{il} A_{m,k} + \Gamma^m_{ik} A_{m,l} + \Gamma^m_{lk} A_{i,m} - \\
- \Gamma^m_{ik,l} A_m + \Gamma^m_{il} \Gamma^p_{mk} A_p + \Gamma^m_{kl} \Gamma^p_{im} A_p + \\
+ \Gamma^m_{ik,l} A_m - \Gamma^m_{ik} \Gamma^p_{ml} A_p - \Gamma^m_{lk} \Gamma^p_{im} A_p = \\
= A_m \left(-\Gamma^m_{ik,l} + \Gamma^p_{il} \Gamma^m_{pk} + \Gamma^p_{kl} \Gamma^m_{ip} + \Gamma^m_{il,k} - \Gamma^p_{ik} \Gamma^m_{pl} - \Gamma^p_{lk} \Gamma^m_{ip}\right) = \\
= A_m \left(-\Gamma^m_{ik,l} + \Gamma^p_{il} \Gamma^m_{pk} + \Gamma^m_{il,k} - \Gamma^p_{ik} \Gamma^m_{pl}\right).
\end{gathered}
$$

Thus

$$
R^m_{ikl} = \Gamma^m_{il,k} - \Gamma^m_{ik,l} + \Gamma^p_{il} \Gamma^m_{pk} - \Gamma^p_{ik} \Gamma^m_{pl}.
$$