From **W. Rindler Special Relativity** Oliver&Boyd 1960

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collision of two moving particles. It will therefore be specified by four coordinates, three of spatial position and one of time, for example (x, y, z, t), if we employ rectangular Cartesian space-coordinates x, y, z. Our investigations will be largely concerned with events. In fact, all physics can be regarded as a study of the pattern of events much as geometry is the study of the pattern of points.

Whether or not two events which are separated in time occur at the same place would seem to be a very simple question. And so it is. Clearly, however, two observers using different frames of reference will not necessarily agree on the matter. Since no one observer can be said to produce the "real" answer we see that spatial position is purely relative and that Newton's premise "space is absolute" must be abandoned. This is a comparatively simple mental adjustment.

Let us now examine the complementary problem, namely how to determine whether two events which are separated in space occur at the same time or not. It had long been taken for granted that, in any given case, the verdict of all competent observers would be unanimous. And yet this is not so. We shall adopt the following practical definition of **simultaneity**: *two events occurring at points P and Q of an inertial frame* $\mathfrak{S}$ *are simultaneous in* $\mathfrak{S}$ *if and only if light emitted at the two events arrives simultaneously at the midpoint of the segment PQ in* $\mathfrak{S}$. This definition is implied by the law of light-propagation of § 6 and it avoids all mention of clocks which would here be an unnecessary complication. Now let $\mathscr{P}$ and $\mathscr{Q}$ be two events occurring simultaneously at points P and Q of an inertial frame $\mathfrak{S}$ and let M be the midpoint of PQ in $\mathfrak{S}$. Let $\mathfrak{S}'$ be a second inertial frame moving in the direction of PQ and let P' and Q' be the fixed points in $\mathfrak{S}'$ at which $\mathscr{P}$ and $\mathscr{Q}$ occur, and let M' be the midpoint of $P'Q'$ in $\mathfrak{S}'$ (see Fig. 1 (*a*); the two figures 1 (*a*) and 1 (*b*) are "snapshots" made in $\mathfrak{S}$). Since $\mathscr{P}$ and $\mathscr{Q}$

are simultaneous in $\mathfrak{S}$, the light-signals from $\mathscr{P}$ and $\mathscr{Q}$ will meet at M. By this time M and M' will have separated owing to the finite velocity of light (Fig. 1 (*b*)). Since the signals cannot meet both at M and at M', it follows that in $\mathfrak{S}'$ the events are not simultaneous. We conclude that *simultaneity at different places is a relative concept*. The now inevitable rejection of Newton's second premise "time is absolute" is a very much more painful mental process than that of his first.

It is the great achievement of Minkowski to have discovered in the wreckage of absolute space and time something which, if perhaps less simple, is nevertheless absolute

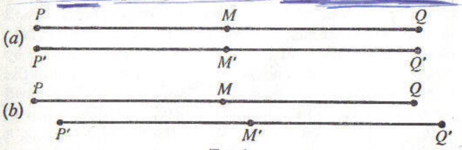

FIG. 1

once more and constitutes a suitable new background for our intuitive thought about the physical world: four-dimensional space-time. This is very much more than a mere matter of terminology, as we shall see in chapter IV.

§ 8. The Lorentz Transformation. In this section we shall consider the transformation of the coordinates of a given event from one inertial frame to another. But as a preliminary we should be quite clear about the method of assigning coordinates to an event in any *one* frame. For this purpose we assume that each observer presiding over an inertial frame is equipped with (i) a **standard clock,** which may be based on any agreed periodic phenomenon, *e.g.* the vibration of the caesium atom (which has actually been used for time measurements), and (ii) a **standard of length,** based, for example, on the wavelength of an

agreed spectral line. By direct measurement, or by means of a base line, two theodolites and an assistant (and Euclidean geometry), the observer can then assign right-handed rectangular Cartesian space-coordinates x, y, z to any event he observes. Knowing the distance of the event and noting the time at which he receives light from it he can, by appeal to the law of light-propagation, also uniquely assign a time-coordinate t to it. Such coordinates (x, y, z, t), which we shall call **standard coordinates,** will be presupposed throughout this book.

In theory it is most convenient to think of the (standard) coordinates of an event as determined locally by auxiliary observers. Once space-coordinates are assigned to all points of the frame, we can imagine identical standard clocks to be placed at the lattice points and observed by auxiliary observers. These clocks can be identified with the free particles defining the frame. They can be synchronized by a control signal emitted, let us say, from the origin at time t_0 by the origin clock. When the signal arrives at a clock whose distance from the origin is r, that clock must be set to indicate time t_0+r/c. On the classical theory this process would evidently synchronize all the stationary clocks of the frame so that equal pointer readings of any two of them always constitute simultaneous events in the sense of the definition of § 7. Now none of the relevant classical laws, in particular the law of light-propagation concerning fixed sources and observers, is affected by relativity. Consequently in relativity, too, the process is a valid one for clock synchronization.† Our imaginary

† It should be noted that, although the light-signalling method is the one usually described for clock synchronization, we could theoretically synchronize the clocks in the frame by purely mechanical means: *e.g.*, by projecting standard particles from standard guns in all directions from a given point. The speed of such particles could be previously determined by projecting one from a point A to a point B whereupon a second must at once be projected back from B to A. The speed sought is evidently twice the distance AB divided by the time elapsed at A.

coordinate lattice carrying auxiliary clocks and observers now allows the space- and time-coordinates (x, y, z, t) of any event to be determined locally.

Let us consider two such frames, $\mathfrak{S}$ and $\mathfrak{S}'$, in uniform relative motion. Let the standard coordinates x, y, z, t in $\mathfrak{S}$ and x', y', z', t' in $\mathfrak{S}'$ be chosen in such a way that (i) $\mathfrak{S}'$ moves in the direction of the positive x-axis of $\mathfrak{S}$ with constant velocity v; (ii) the two x-axes and their

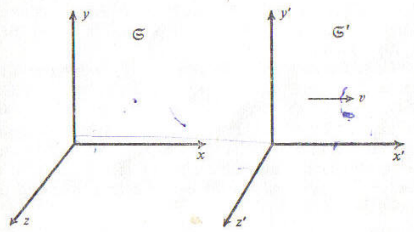

FIG. 2

positive senses coincide; (iii) the coordinate planes $y = 0$ and $z = 0$ coincide permanently with the coordinate planes $y' = 0$ and $z' = 0$, respectively; and (iv) the two spatial origins coincide when their local clocks both read zero. We shall in future call this the **standard configuration** of two frames $\mathfrak{S}$ and $\mathfrak{S}'$ (Fig. 2). Outside of classical mechanics the feasibility of stipulations (ii) and (iii) needs justification. We return to this point below (on p. 17); till then the argument is independent of the configuration.

If (x, y, z, t) and (x', y', z', t') are the coordinates in $\mathfrak{S}$ and $\mathfrak{S}'$ respectively of an arbitrary event, our problem is to find the relations between these two sets of numbers.

What make it sure that both observer can agree what is the same event? Space-time absolute!? ~ I cf p 13

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The simple so-called **Galilean transformation,**

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t, \tag{1.1}$$

which is valid in Newtonian mechanics, is not in accordance with our result that simultaneity is relative. Moreover we cannot remedy this defect by a mere amendment of the last member, for a simple consideration shows that the transformation of the x-coordinate is affected by the same objection (see § 10, penultimate paragraph). We shall therefore derive the required transformation equations afresh by appeal to the relativity principle and the law of light-propagation.

Consider any event $\mathscr{P}$ and a neighbouring event $\mathscr{Q}$ (close to $\mathscr{P}$ in $\mathfrak{S}$ *and* $\mathfrak{S}'$) whose coordinates differ from those of $\mathscr{P}$ by dx, dy, dz, dt in $\mathfrak{S}$ and by dx', dy', dz', dt' in $\mathfrak{S}'$. Suppose that at the event $\mathscr{P}$ a flash of light is emitted and that $\mathscr{Q}$ is the event of some particle in space being illuminated by that flash. In accordance with the law of light-propagation the observer in $\mathfrak{S}$ will find that $(dx^2+dy^2+dz^2)^{\frac{1}{2}} = cdt$, or

$$dx^2+dy^2+dz^2-c^2dt^2 = 0, \quad dt>0, \tag{1.2}$$

and, similarly, the observer in $\mathfrak{S}'$ will find that

$$dx'^2+dy'^2+dz'^2-c^2dt'^2 = 0, \quad dt'>0. \tag{1.3}$$

Conversely, any event near $\mathscr{P}$ whose coordinates satisfy *either* (1.2) *or* (1.3) is illuminated by the flash from $\mathscr{P}$ and therefore its coordinates will satisfy *both* (1.2) *and* (1.3). Now, no matter what the transformations between the coordinates themselves may be, provided they are differentiable, the transformations between the differentials at any fixed event $\mathscr{P}$ are linear and homogeneous (as always) and thus the left member of (1.3) equals a homogeneous quadratic in dx, dy, dz, dt. This quadratic, as we have just seen, must vanish for all real values of the differentials which satisfy (1.2). It can easily be shown (see exercise

$(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2=c^2(t_2-t_1)^2 \quad t_2>t_1$

I (1), p. 21) that it must therefore be a multiple of the quadratic in (1.2). And since only the *ratios* of the differentials matter here, we have introduced no restriction by confining our attention to an infinitesimal neighbourhood of $\mathscr{P}$. Thus at *any* event $\mathscr{P}$ the following relation holds:

$$dx^2+dy^2+dz^2-c^2dt^2 = K(dx'^2+dy'^2+dz'^2-c^2dt'^2), \tag{1.4}$$

where K is independent of the differentials. Furthermore, K at $\mathscr{P}$ is independent of the choice of standard coordinates in $\mathfrak{S}$ and $\mathfrak{S}'$. For, since the frames are Euclidean, the values of $dx^2+dy^2+dz^2$ and $dx'^2+dy'^2+dz'^2$ relevant to $\mathscr{P}$ and $\mathscr{Q}$ are independent of the choice of axes, and by the homogeneity of time the values of dt^2 and dt'^2 are independent of the choice of the origins of time. Without affecting the value of K at $\mathscr{P}$ we can therefore choose coordinates so that $\mathscr{P} = (0, 0, 0, 0)$ in $\mathfrak{S}$ and $\mathfrak{S}'$. Since the orientations of the rectangular axes in $\mathfrak{S}$ and $\mathfrak{S}'$ can be arbitrary for the present argument, and since inertial frames are isotropic, the relation of $\mathfrak{S}$ and $\mathfrak{S}'$ relative to each other and to the event $\mathscr{P}$ is now completely symmetric whence we must have, as well as (1.4),

$$dx'^2+dy'^2+dz'^2-c^2dt'^2 = K(dx^2+dy^2+dz^2-c^2dt^2).$$

It follows that $K = \pm 1$. $K = -1$ can at once be dismissed, since (1.4) must remain valid as $v \to 0$. Conquently,

$$dx^2+dy^2+dz^2-c^2dt^2 = dx'^2+dy'^2+dz'^2-c^2dt'^2 \tag{1.5}$$

for differentials at $\mathscr{P}$ and evidently at all other events too.

Equation (1.5) implies that the transformation equations between the primed and unprimed coordinates must be linear. (For a proof, see exercise IV (1), p. 74. The proof is postponed only because the most convenient notation for it is not introduced until chapter IV. See also exercise I (2), p. 21.)

The linearity of the transformation implies that the coordinate axes can indeed be oriented to give the

"standard configuration" mentioned above. In $\mathfrak{S}$ consider a fixed plane with equation $lx+my+nz+p = 0$. In $\mathfrak{S}'$ this becomes, say $l(a_1x'+b_1y'+c_1z'+d_1t'+e_1)$ $+m(a_2x'+\ldots)+n(a_3x'+\ldots)+p = 0$, which represents a moving plane unless $ld_1+md_2+nd_3 = 0$, *i.e.*, unless the normal vector (l, m, n) to the plane in $\mathfrak{S}$ is perpendicular to the vector (d_1, d_2, d_3). All such planes evidently intersect in lines which are fixed in both $\mathfrak{S}$ and $\mathfrak{S}'$, and which are parallel to the vector (d_1, d_2, d_3) in $\mathfrak{S}$. These lines must correspond to the direction of relative motion of the frames. By symmetry, two such planes which are orthogonal in $\mathfrak{S}$ must also be orthogonal in $\mathfrak{S}'$. This allows the choice of the two common coordinate planes.

Under a linear transformation the finite coordinate differences satisfy the same transformation equations as the differentials. It therefore follows from (1.5) when applied to the event (0, 0, 0, 0) that, for any event with coordinates (x, y, z, t) in $\mathfrak{S}$ and (x', y', z', t') in $\mathfrak{S}'$, the following relation holds:

$$x^2+y^2+z^2-c^2t^2 = x'^2+y'^2+z'^2-c^2t'^2. \tag{1.6}$$

Now, by hypothesis, the coordinate planes $y = 0$ and $y' = 0$ coincide permanently. Thus $y = 0$ must imply $y' = 0$, whence we can set

$$y' = Ay, \tag{1.7}$$

where A is a constant (possibly depending on v). By reversing the directions of the x- and z-axes in $\mathfrak{S}$ and $\mathfrak{S}'$ we can interchange the roles of these frames (presupposing isotropy as in the argument for K) without affecting (1.7); but then, by symmetry, we also have

$$y = Ay',$$

whence $A = \pm 1$. The negative sign can again be dismissed since $v \to 0$ must imply $y' \sim y$, and so $A = 1$. The argument for z is similar, whence we have

$$y' = y, \quad z' = z, \tag{1.8}$$

as in the Galilean case.

In virtue of (1.8), equation (1.6) now reduces to

$$x^2-c^2t^2 = x'^2-c^2t'^2. \tag{1.9}$$

Since $x' = 0$ must imply $x = vt$, we can set

$$x' = B(x-vt),$$

where B is a constant (possibly depending on v). From this and (1.9) it follows that t' is of the form

$$t' = Cx+Dt,$$

where C and D are constants (possibly depending on v). When these expressions for x' and t' are substituted in (1.9), and the three equations that result from comparing the coefficients of x^2, xt, t^2 are solved, we find

$$B = D = \frac{1}{\pm(1-v^2/c^2)^{\frac{1}{2}}}, \quad C = \frac{-v/c^2}{\pm(1-v^2/c^2)^{\frac{1}{2}}},$$

where again we must choose the positive sign for the same reason as before. Thus, collecting our results, we have obtained the transformation equations

$$x' = \frac{x-vt}{(1-v^2/c^2)^{\frac{1}{2}}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t-vx/c^2}{(1-v^2/c^2)^{\frac{1}{2}}}, \tag{1.10}$$

which are usually called the **Lorentz equations.** If the relativity principle is true, then *all the laws of physics which are valid in an inertial frame must be invariant under these transformation equations.* We proceed to list some of their more important properties:

(i) The Lorentz equations replace the older Galilean equations (1.1), to which they nevertheless approximate when v is sufficiently small. (For example, $(1-v^2/c^2)^{-\frac{1}{2}} < 1{\cdot}01$ as long as $v < \frac{1}{7}c$, at which speed the earth is circled in one second.) This is in agreement with the high degree of accuracy with which Newtonian mechanics (invariant under the Galilean transformation) describes a large domain of nature.

(ii) We can see how intimately the difference between the Galilean and the Lorentz equations is connected with the finite speed of light by letting $c \to \infty$, when (1.10) goes over into (1.1).

(iii) When $v = c$, two members of (1.10) become infinite, and $v > c$ leads to imaginary values. This is the first indication of a fact we shall examine more closely in the next chapter, namely that the relative velocity between inertial frames cannot exceed the speed of light.

(iv) The appearance of the space-coordinate x in the transformation of the time is the mathematical expression of the relativity of simultaneity. It implies that two events corresponding to equal values of t do not necessarily correspond to equal values of t'.

(v) Equations (1.10) are symmetric not only in y and z but also in x and ct. (The reader should verify this by writing T/c for t and T'/c for t' in (1.10) and multiplying the last equation by c.) In the sequel we shall often find ct a more convenient variable than t.

(vi) The Lorentz transformations are non-singular (their determinant is easily seen to be unity) and they possess the two so-called group properties.† First, direct algebraic

† The requirements for an abstract multiplicative group are (i) the product of two elements is an element of the group; (ii) the associative law $(ab)c = a(bc)$ holds; (iii) there is a unit element e satisfying $ae = ea = a$ for all a; (iv) each element a has an inverse a^{-1} such that $a^{-1}a = aa^{-1} = e$.

A *transformation group* is a set of transformations which form a group in which the product of two transformations is their resultant transformation, the unit element is the identity transformation and inverse elements are inverse transformations in the usual sense. Now any set of transformations is associative; if the set is non-void and (*a*) the resultant of any two transformations of the set is in the set and (*b*) the inverse of each transformation of the set is in the set, it follows that the identity transformation is in the set and that the set forms a transformation group. For this reason (*a*) and (*b*) are called the *group properties*. The only explicit use of group theory in this book is to provide this name for (*a*) and (*b*).

solution of (1.10) for x, y, z, t gives

$$x = \frac{x' + vt'}{(1 - v^2/c^2)^{\frac{1}{2}}}, \quad y = y', \quad z = z', \quad t = \frac{t' + vx'/c^2}{(1 - v^2/c^2)^{\frac{1}{2}}}, \qquad (1.11)$$

and thus the inverse of (1.10) is a Lorentz transformation with parameter $-v$ instead of v, as must indeed be the case from symmetry considerations. Second, the resultant of two successive Lorentz transformations, with parameters v_1 and v_2 respectively, is also found to be of type (1.10) with parameter $v = (v_1 + v_2)/(1 + v_1 v_2/c^2)$.

We note, finally, that any effect whose speed of propagation *in vacuo* is finite and constant could have been used, as light was, in the derivation of the Lorentz equations. Since only one transformation can be valid, it follows that all such effects must be propagated with the speed of light. Examples are provided by electromagnetic waves of all frequencies.

Exercises I

(Unless otherwise indicated, two frames $\mathfrak{S}$ and $\mathfrak{S}'$ will always be understood to be in standard configuration.)

(1) Prove that if the polynomial

$$P \equiv aX^2 + bY^2 + cZ^2 + dT^2 + gXT + hYT + kZT + lYZ + mXZ + nXY$$

vanishes whenever the polynomial

$$Q \equiv X^2 + Y^2 + Z^2 - T^2$$

vanishes for real X, Y, Z, T and $T > 0$, then P can differ from Q by at most a constant factor. [Hint: substitute into P in turn the following obvious zeros of Q: $(\pm 1, 0, 0, 1)$, $(0, \pm 1, 0, 1)$, $(0, 0, \pm 1, 1)$, $(0, 1/\sqrt{2}, 1/\sqrt{2}, 1)$, $(1/\sqrt{2}, 0, 1/\sqrt{2}, 1)$, $(1/\sqrt{2}, 1/\sqrt{2}, 0, 1)$, and solve the resulting conditions on the coefficients.]

(2) For proof of the linearity of the transformation between the standard coordinates in two inertial frames,

B

(För att prova i på (1,2) o. (1,3) där vi menar [illegible] infinitesimala storheter multiplicera [illegible] ovan med ε som → 0)