Algebra 1 – The basics of equation manipulation

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Rev 1 - Revised 11/24/10 to add notes about the distributive law

Purpose: To take kids who've learned arithmetic and get them started into algebra.

Let's go! Okay, you've learned how to add, subtract, multiply, and divide, and done a little in fractions, decimals, squares, square roots and using parentheses to control the order of operations in an equation. You know that an equation is like a sentence that has an = sign in the middle of it, which means that the part on the left equals the part on the right.

They've also told you that a letter can stand for a number, and they have you solve simple equations like:

find *X*, when $X + 3 = 5$

and it doesn't take a lot of head scratching to figure out that *X*=2.

Now you are ready for algebra, and here it is in one sentence:

The purpose of algebra is to manipulate equations until they tell us what we want to know.

The trick is that real equations are full of letters, so we have to have rules for working on equations full of letters. In this paper, we will introduce just one rule:

The Basic Rule of manipulating equations – If you do the same thing to each side of an equation, the equation will still be accurate.

1. You can add the same number or letter to each side of an equation,
2. You can subtract the same number or letter from each side of an equation,
3. You can multiply each side of an equation by the same number or letter,
4. You can divide each side of an equation by the same number or letter,
5. You can square each side of an equation,
6. You can take the square root of each side of an equation,
7. etc.,

and the equation will still be accurate.

Let's give it a try on the equation above. We'll do it step by step.

a) Start with $X + 3 = 5$

b) Subtract 3 from both sides $X + 3 - 3 = 5 - 3$

c) Result $X = 2$

That's pretty simple. Now let's try one that has a multiplication in it.

a) Start with $2X = 10$

b) Divide each side by 2 $\frac{2X}{2} = \frac{10}{2}$

c) Result $X = 5$

Note that we write $2 \times X$ as $2X$ and $10 \div 2$ as $\frac{10}{2}$

These are the cool ways to write multiplication and division. Sometimes a multiplication sign is used (∗ or maybe × but not usually •) when it makes the equation clearer, but a division is always written like a fraction. Put away that old ÷ sign. It's kid stuff. If you want to write a division or a fraction on one line, use the forward slash, such as 10 / 2.

So how do we know what to do to an equation? Well, if you look at the two examples above, what we wanted for the result was the variable *X* on one side of the equation and a number on the other side. We picked numbers to subtract or divide on both sides that would clear out the numbers from the left side and just leave the letter *X*. 3 – 3 = 0 and that clears out a number that was added to *X*. 2 / 2 = 1 and that clears out a number that was multiplied by *X*. Pretty simple. You should look at manipulating equations like it's a game and have fun doing it.

OK, let's try one that has both a multiplication and an addition.

a) Start with $4X + 7 = 43$

b) Step 1: Subtract 7 from both sides $4X + 7 - 7 = 43 - 7$

c) First result $4X = 36$

d) Step 2: Divide both sides by 4 $\frac{4X}{4} = \frac{36}{4}$

e) Final result $X = 9$

That equation didn't stand a chance. We picked it apart in two steps.

Now let's pick an equation that has a subtraction and a division.

a) Start with $\frac{48}{X} - 11 = 5$

b) Step 1: Add 11 to both sides $\frac{48}{X} - 11 + 11 = 5 + 11$

c) Result of step 1 $\frac{48}{X} = 16$

d) Step 2: Multiply both sides by X $\frac{48X}{X} = 16X$

e) Result of step 2 $48 = 16X$

f) Step 3: Divide both sides by 16 $\frac{48}{16} = \frac{16X}{16}$

g) Final result $3 = X$ or $X = 3$

There we go. Step by step we move the numbers and letters around to separate them and get the answer we're looking for. Each step is simple and each step makes the equation simpler.

OK, here's a few for you to try. Write out each step as we did in the examples above. You can use this paper or clean sheets of paper. If you are using a loose leaf notebook to hold this paper, add your pages right after the last page and number them 5A, 5B, 5C, etc. so you can keep them in the right order.

Try 1 (takes 2 steps and $X = 8$) $5X + 11 = 51$

Try 2 (takes 2 steps and $X = 14$) $\frac{X}{2} + 6 = 13$

Try 3 (takes 3 steps and you figure out the answer) $\frac{24}{X} - 3 = 3$

If you are having trouble, you should make up some more problems at this point and work them out until you are good at it. Then proceed on.

All right, you've seen some simple problems. The same rules can be applied to more complicated equations. Let's look at a couple:

a) Find X, starting with $\frac{3X^2 + 5}{12} = 9$

b) Step 1 – multiply each side by 12 $\frac{(3X^2 + 5)12}{12} = 9 * 12$

c) Result of step 1 $3X^2 + 5 = 108$

d) Step 2 – Subtract 5 from each side $3X^2 + 5 - 5 = 108 - 5$

e) Result of step 2 $3X^2 = 103$

f) Step 3 – Divide each side by 3 $\frac{3X^2}{3} = \frac{103}{3}$

g) Result of step 3 $X^2 = 34.333$

h) Step 4 – Take the square root of each side $\sqrt{X^2} = \sqrt{34.333}$

i) Result of step 4 $X = 5.859$

OK, that was fun. Let's try another one.

a) Find *X*, starting with $\frac{A}{X} = \frac{B}{C}$

b) Step 1 – Multiply each side by *X* $\frac{AX}{X} = \frac{BX}{C}$

c) Result of step 1 $A = \frac{BX}{C}$

d) Step 2 – Multiply each side by *C* $AC = \frac{BXC}{C}$

Notice that when we want to say two letters are multiplied by each other we just write them side by side, without any multiplication sign between them. That's the cool way to do it. We can also show multiplication between expressions in parentheses by writing them side by side without a multiplication sign, like $(A + B)(C + D)$

e) Result of step 2 $AC = BX$

f) Step 3 – Divide each side by *B* $\frac{AC}{B} = \frac{BX}{B}$

g) Result of step 3 $\frac{AC}{B} = X$ or $X = \frac{AC}{B}$

Yeah, you can handle unknowns like *A*, *B*, and *C* just like numbers. Just for fun, try this next one and compare it with the one above. It takes only one step:

Find *X*, starting with $\frac{X}{A} = \frac{B}{C}$

Now try an interesting-looking one: $\sqrt{\frac{X+4}{5}} = 3$

Hint: The first step should be to square both sides of the equation. Remember that

$$\left(\sqrt{A}\right)^2 = A$$

All right, we've had enough fun for one paper. You are now qualified to perform algebraic manipulations on equations using the Basic Rule on the first page above. Go forth and algebrate!

One last question – can we write equations that we can't solve? You bet we can.

Try to solve this equation for *X*: $X + \frac{1}{X} = 0$

After 3 steps you get $X = \sqrt{-1}$ which is not a real number.

Now try to solve $X^2 - 3X + 2 = 0$ for *X*. The simple steps of the Basic Rule do not provide an obvious way to solve this equation. So, be happy! There is more algebra to be written in a later paper!

By the way, it turns out that the last equation above has *two* solutions, *X*=1 and *X*=2. Take these numbers one at a time and substitute them for the *X*s in the equation and see for yourself.

Note from the field 11/24/10 – The biggest problem students have in manipulating equations has proven to be the application of the distributive law:

$$A(B + C) = AB + AC$$

This might seem a simple concept to an adult, but 6th, 7th, and even 8th graders can have a lot of trouble with it. Solution of more complicated equations often requires expanding expressions in parentheses or gathering terms into parentheses and bringing out a common factor. Here is an example:

a) Solve for X $\quad X^2 - 3X = \frac{2X}{3}$

b) use distributive law on the left side $\quad X(X - 3) = \frac{2X}{3}$

c) divide each side by X $\quad X - 3 = \frac{2}{3}$

d) multiply each side by 3 $\quad 3(X - 3) = 2$

e) use the distributive law on the left side $\quad 3X - 9 = 2$

f) add 9 to each side $\quad 3X = 11$

g) divide each side by 3 $X = \frac{11}{3}$

Make up problems that involve the distributive law until the student is proficient with it.

Note for parents or other persons who have accepted the responsibility of a teacher: As kids mature, their ability to understand complex ideas matures. It might take your kids a while to wrap their minds around the ideas in this topic, and you will have to be their guide. The younger a kid is, the more he or she learns through the ears rather than the eyes. Don't be shy about inventing different ways to pour these concepts into your kids' ears. And make up as many sample problems as you need. Remember that learning in the brain is biological, not electrical, and it takes time for the brain to grow the structures that can crank out a new skill. Print these sheets, let kids write all over them, and date them and keep them in a notebook, so they can one day look back over the journey they've taken.