Lecture 7

Energy transport by radiation

The complete treatment of the transport of energy by radiation, and the interaction between radiation and matter, is a research topic in its own right, which has been extensively treated in the literature (e.g. Mihalas D., 1978, Stellar Atmospheres, 2nd ed., W. H. Freeman, San Francisco). Such a complete treatment is required to analyze the properties of, for example, stellar atmospheres or the interstellar medium. In stellar interiors, however, it is possible to get by with a simplified description whose results agree with those of the more complete theory in the limit where the *mean free path* of the photons is very short.

7.1. Mean free path and opacity

The mean free path of a photon depends on a microscopic interaction between radiation and matter. Traditionally this interaction is described in terms of a cross section σ_R, such that, on average, a photon interacts with an atom if it passes within the area σ_R around the atom. If the number of atoms per unit volume is n , the mean free path λ_{ph} of a photon is such that

$$\lambda_{ph}\sigma_R = \text{volume per unit atom} = \frac{1}{n}.$$

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(7.1)

Instead of using λ_{ph} to describe the interaction between matter and radiation it is conventional, and convenient, to use the opacity κ , defined as

$$\kappa = \frac{1}{\rho\lambda_{ph}} = \frac{n}{\rho}\sigma_R.$$

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(7.2)

Note that n/ρ is the number of atoms per unit mass; hence κ is the total cross section per unit mass. If σ_R and n/ρ is independent on the state of the gas (as described by its density and temperature), it follows that κ is also independent on ρ and T.

Typical values for the opacity for stellar material are of order $\kappa \approx 1cm^2/g$. with Typical values of density of order $\rho \approx 1g/cm^3$, the typical values of the mean free path are $\lambda_{ph} \approx 1cm$, i.e. stellar matter is very opaque. It follows that for radiative transport in stars, the mean free path λ_{ph} of the “transporting particles” (photons) is very small compared to the characteristic length R (stellar radius) over which the transport extends. In this case, the transport can be treated as a *diffusion process*, which yields an enormous simplification of the problem.

7.2. The diffusion approximation

In this section we show that the diffusive flux $\mathbf{j}$ of “transporting particles” (per unit area and time) between places of different particle density n is given by

$$\mathbf{j} = -D\nabla n,$$

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where D is the coefficient of diffusion,

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determined by the average values of mean velocity v and mean free path λ of the particles. This result is, of course, of much wider applicability, not limited by the problem of radiation diffusion.

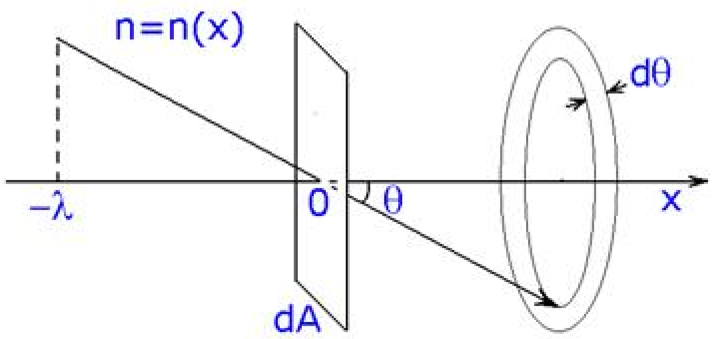

Figure 7.1. Particles passing through the plane at $x=0$ at the angle θ with the normal originate, on average, at coordinate $x=-\lambda\cos\theta$, where λ is the mean free path.

We consider the transport of particles in the time interval dt through an area dA orthogonal to axis x , at $x=0$ (Fig. 7.1). We choose axis x in the direction of ∇n, so that $n=n(x)$ in the vicinity of $x=0$, and $dn/dx>0$. We describe the direction of motion of particles by the angle θ with axis x. The motion of the particles is assumed to be almost isotropically distributed in direction; then out of the total number of particles, the fraction of particles with directions between θ and $\theta+d\theta$ is $2\pi\sin\theta d\theta/4\pi=(1/2)\sin\theta d\theta$.

We first consider particles that go through dA with directions between θ and $\theta+d\theta$. On average, they come from the distance $\lambda\cos\theta$ from the plane, where the particle density is $n(x)=n(-\lambda\cos\theta)$. Their contribution to the flux of particles through dA is

$$\underbrace{\frac{1}{2}\sin\theta\, d\theta \cdot n(-\lambda\cos\theta)\cdot v\,dt}_{\text{Particle flux through unit area orthogonal to propagation direction}} \times \underbrace{dA\cdot\cos\theta}_{\text{Projected area}}$$

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(7.5)

The net number of particles dN passind through dA from the left to the right during time interval dt is obtained by integrating over all directions:

$$
\begin{aligned}
dN &= \frac{1}{2} v\, dt\, dA \int_0^{\pi} n(-\lambda \cos\theta) \cdot \cos\theta \sin\theta \, d\theta \\
&\simeq \frac{1}{2} v\, dt\, dA \int_0^{\pi} n(0) \cdot \cos\theta \sin\theta \, d\theta \\
&- \frac{1}{2} v\lambda \frac{dn}{dx} dt\, dA \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta \\
&= -\frac{v\lambda}{3} \frac{dn}{dx} dA\, dt,
\end{aligned}
$$

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&- \frac{1}{2} v\lambda \frac{dn}{dx} dt\, dA \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta \\
&= -\frac{v\lambda}{3} \frac{dn}{dx} dA\, dt,
\end{aligned}
\tag{7.6}
$$

where we used a Taylor expansion of $n(x)$ in x , which is $n(x) \cong n(0) + x(dn/dx)_{x=0}$. Comparing with equation (7.3), we have $|j| = dN/dA/dt$ and $|\nabla n| = dn/dx$, and thus the diffusion coefficient D is given by the equation (7.4).

7.3. Equation of radiative transport

We now apply the diffusion approximation to the energy transport by radiation. In order to obtain the diffusive flux of radiative energy $\mathbf{F}$, we replace the number density of transporting particles by the energy density of radiation u_R, mean velocity v by the velocity of light c , and λ by $\lambda_{ph}=1/(\kappa\rho)$ (equation 7.2). Owing to the spherical symmetry of the problem, $\mathbf{F}$ has only a radial component $F_r=|F|=F$, and ∇n reduces to the derivative in radial direction, du_R/dr. Then equations (7.3, 7.4) give immediately that

$$F = -\frac{1}{3} c\lambda_{ph} \frac{du_R}{dr} = -\frac{1}{3} \frac{c}{\kappa\rho} \frac{du_R}{dr}.$$

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(7.7)

To evaluate the radiative energy flux F, we now only need to specify the energy density of the radiation.

The radiation energy density u_R in deep stellar interior is described in the black body approximation. The black body is a perfect absorber and a perfect radiator. If the radiation is in thermodynamic equilibrium with its surroundings, as, for example, in an adiabatic enclosure whose walls are maintained at a constant temperature T, each unit area of the surface emits as much radiant energy as it absorbs in each frequency, and the conditions of black-body radiation are fulfilled.

The *specific intensity* I_ν of the black-body emission (the amount of energy per unit frequency interval and per unit time, flowing through unit area in unit solid angle) is given by the Planck function

$$I_\nu = B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{\exp(h\nu/kT)-1},$$

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(7.8)

where ν is frequency $(\nu=c/\lambda$, where λ is wavelength), h is Planck's constant, k is Boltzman's constant, and c is speed of light.

We have already used the Planck function in Lecture 2, in discussing stellar luminosity (equations 2.9-2.13). Indeed, using the geometry shown in Figure 7.1, the energy emitted in unit time by unit area of a black body in all the directions and in all the frequencies is

$$\int_{\nu=0}^{\infty} d\nu \int_{\theta=0}^{\pi/2} B_\nu \cos\theta \cdot 2\pi \sin\theta \, d\theta$$

$$= \pi \int_{\nu=0}^{\infty} B_\nu \, d\nu = \sigma T^4,$$

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(7.9)

which gives $L = 4\pi R^2 \sigma T^4$ (i.e. equation 2.13), where

$$\sigma = \frac{2\pi^5 k^4}{15 c^2 h^3}$$

$$\sigma = \frac{2\pi^5 k^4}{15 c^2 h^3}$$ (7.10)

is Stefan-Boltzman constant (the last expression can be obtained with using $\int_0^\infty x^3 dx/(e^x-1) = \pi^4/15$).

We now need a relation between integrated radiation intensity

$$I = \int_0^\infty I_\nu \, d\nu = \int_0^\infty B_\nu \, d\nu = \frac{\sigma}{\pi} T^4$$

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(7.11)

and the energy density of the radiation u_R. To get this relation, consider the amount of energy crossing small area ds in solid angle $d\omega$ in time dt; this is $I \, ds \, d\omega \, dt$. This energy occupies the volume $ds \, c \, dt$. The energy density of the radiation propagating in a solid angle $d\omega$ is thus

$$du_R = \frac{I\,ds\,d\omega\,dt}{ds\,c\,dt} = \frac{I}{c}d\omega\,.$$

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(7.12)

The total energy density is obtained by integrating over all the solid angles, $\int d\omega = 4\pi$:

$$u_R = 4\pi\frac{I}{c} = \frac{4\sigma}{c}T^4,$$

$$u_R = 4\pi\frac{I}{c} = \frac{4\sigma}{c}T^4,$$

(7.13)

using equation (7.11). Introducing

$$a = \frac{4\sigma}{c} = \frac{8\pi^5 k^4}{15\,c^3 h^3},$$

$$a = \frac{4\sigma}{c} = \frac{8\pi^5 k^4}{15\,c^3 h^3},$$

(7.14)

known as Stefan radiation constant, $a = 7.56 \times 10^{-16}\ \ Jm^{-3}K^{-4}$, we have

$$u_R = aT^4,$$

$$u_R = aT^4,$$

(7.15)

the expression for the radiation energy density which we need in our analysis.

Going back to the equation (7.7) for the diffusive flux of radiative energy F , we get

$$F = -\frac{4acT^3}{3\kappa\rho}\frac{dT}{dr}.$$

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(7.16)

If the energy transport occurs only through radiation, the total amount of energy transported through a sphere of radius r in unit time is

$$L(r) = 4\pi r^2 F.$$

$$L(r) = 4\pi r^2 F.$$

(7.17)

We therefore finally arrive to the equation

$$\frac{dT}{dr} = -\frac{3\kappa\rho L(r)}{16\pi a c r^2 T^3},$$

(7.18)

$$\frac{dT}{dr} = -\frac{3\kappa\rho L(r)}{16\pi a c r^2 T^3},$$

which relates the temperature gradient with $L(r)$, one of the fundamental equations of stellar structure.

7.4. Opacity in stellar interiors.

The computation of the cross section σ_R is in general a very complicated numerical problem, where account must be taken of the detailed interaction between the radiation and the different atoms in the gas. Hence it is common in computations of stellar models to use tables over the dependence of opacity on ρ , T and the chemical composition. However, there are simple approximations which give a feel for the dependence of the opacity on the thermodynamical state.

The opacity arising from the interaction between radiation and atoms can approximately be expressed as

$$\kappa = 4.3 \times 10^{25} Z(1+X)\rho T^{-3.5} \text{ cm}^2/\text{g},$$

$$\kappa = 4.3 \times 10^{25} Z(1+X)\rho T^{-3.5} \text{ cm}^2/\text{g},$$ (7.19)

the so-called *Kramers approximation*. This contribution dominates in the interior of relatively light stars, where the temperature is relatively low. At higher temperature, i.e. in more massive stars, scattering off free electrons dominates. The cross section σ_R for this process is independent of ρ and T ; the same is true of the number n_e/ρ of electrons per unit mass, if we assume that the gas is completely ionized. Hence the opacity is also independent on ρ and T (see equation 7.2); one finds that

$$\kappa = 0.2(1+X)\text{ cm}^2/\text{g}.$$

$$\kappa = 0.2(1+X)\text{ cm}^2/\text{g}.$$ (7.20)

7.5. The main sequence.

We can estimate the luminosity of stars from the equation of radiative transport, combined with our previous estimates of the temperature and density in stars. As usual, the purpose is to get a feeling, within a few orders of magnitude, for the characteristic value of the luminosity, and an idea about how it varies with the parameters characterizing the star. Hence in general we neglect factors of order unity.

We assume the ideal gas law; then we have the estimate for the temperature

$$T \approx \frac{GMm_H \mu}{kR}$$

$$T \approx \frac{GMm_H \mu}{kR}$$

(7.21)

(equation 4.10), and we estimate the density by mean density

$$\rho \approx \frac{M}{R^3}.$$

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(7.22)

The luminosity is determined by the equation of radiative transport (equation 7.18), which we write as

$$L = -\frac{4\pi r^2 ac}{3\kappa\rho}\frac{dT^4}{dr}.$$

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(7.23)

We approximate the opacity by a power law

$$\kappa = \kappa_0 \rho^{\lambda} T^{-\nu}$$

$$\kappa = \kappa_0 \rho^{\lambda} T^{-\nu}$$

(7.24)

(cf. equations 7.19 and 7.20). Finally we replace r by R , and approximate $-dT^4/dr$ by T^4/R . Then we obtain

$$L \approx \frac{acRT^{4+\nu}}{\kappa_0 \rho^{\lambda+1}} \approx \frac{ac}{\kappa_0}\left(\frac{Gm_H\mu}{k}\right)^{4+\nu} R^{3\lambda-\nu} M^{3+\nu-\lambda}.$$

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(7.25)

It may seem peculiar that we can calculate the stellar luminosities without taking into account the processes that are responsible for energy generation. The explanation is that the star is in equilibrium, so that all parts of the star have to “fit together”. The energy production has to adjust itself to produce the amount of energy necessary to satisfy the equation (7.25). This is possible because the rate of energy production is a very sensitive function of temperature, as shown in Lecture 6. Hence a small modification of the central temperature is sufficient to obtain the correct luminosity.

But we can also estimate L using the energy generation rates. Comparison of the two results allows to establish simple scaling relations which describe the location of the main-sequence stars on the Hertzsprung-Russel diagram. We distinguish between two cases:

(1) *Lower main-sequence, relatively low masses*.

Here the temperature is relatively low, and the opacity is dominated by atomic processes, in particular bound-free transitions. Hence the opacity can be approximated by the Kramers law, i.e. $\lambda = 1$, $\nu = 3.5$. Then for stars of nearly the same chemical composition, equation (7.25) gives

$$L \propto M^{5.5} R^{-0.5}.$$

$$L \propto M^{5.5} R^{-0.5}.$$

(7.26)

The energy generation is dominated by the PP chain. For stars of nearly the same polytropic index, equation (6.10) gives, with using $a = 4.5$,

$$L \propto M^{6.5} R^{-7.5}.$$

$$L \propto M^{6.5} R^{-7.5}.$$

(7.27)

We also have a relation between stellar luminosity and its effective temperature (equation 2.13), which gives

$$L \propto T_{eff}^4 R^2.$$ $$L \propto T_{eff}^4 R^2.$$ (7.28)

From these three relations, we have

$$R \propto M^{1/7}, \quad L \propto M^{38/7}, \quad T_{eff} \propto M^{9/7}.$$

$$R \propto M^{1/7}, \quad L \propto M^{38/7}, \quad T_{eff} \propto M^{9/7}.$$ (7.29)

Relation between L and T_{eff}, represented by the HR diagram, is thus

$$L \propto T_{eff}^{38/9}.$$ $$L \propto T_{eff}^{38/9}.$$ (7.30)

It predicts that the stars of the lower main sequence shall be represented by a straight line in the $\log L$ - $\log T$ coordinates of the HR diagram, with slope of about 4.

(2) *Upper main sequence, relatively massive stars*.

Here the temperature is relatively high, and the opacity is dominated by the electron scattering. In the equation (7.25), we have $\lambda = \nu = 0$, and hence

$$L \propto M^3.$$ $$L \propto M^3.$$ (7.31)

The energy generation for massive main-sequence stars is dominated by CNO cycle. The energy generation rate per unit mass ε is roughly proportional to ρT^{16} for stars of similar chemical composition (equation 6.11). We can thus use the same equation (6.10) for the luminosity, but now with $a = 16$, getting

$$L \propto M^{18} R^{-19}.$$ $$L \propto M^{18} R^{-19}.$$
(7.32)

Equation (7.28) remains the same, and we now get

$$R \propto M^{15/19}, \quad L \propto M^{3}, \quad T_{eff} \propto M^{27/76}.$$

$$R \propto M^{15/19}, \quad L \propto M^{3}, \quad T_{eff} \propto M^{27/76}. \qquad (7.33)$$

A required relation between L and T_{eff} is now

$$L \propto T_{eff}^{76/9}.$$

$$L \propto T_{eff}^{76/9}. \qquad (7.34)$$

The predicted slope of the the $\log L - \log T$ relation on the HR diagram is now about 8.5 for the upper main sequence, about twice steeper than for the lower main sequence.

The scaling relations developed in this section are, of course, very rough---they are based on the simplest order-of-magnitude estimates for the luminosity (equation 7.25), provided by the equation of radiative energy transport (7.18). Also, as we will see in the next Lecture, radiation is not the only means of energy transport in stars. Nevertheless, these results are not very far from observations, and allow to get a good feeling about the origin of the well-defined main-sequence domain in the HR diagrams, without going into the extensive numerical computations.

Exercises

Exercise 7.1. A group of stars all have the same chemical composition, opacity dominated by electron scattering $\kappa = 0.2(1+X)\ \mathrm{cm^2/g}$, and energy generation rate per unit mass $\varepsilon = \varepsilon_0 \rho T^{18}$ with some constant ε_0. Using simplest order-of-magnitude considerations, estimate the dependence of R and L on stellar mass M for this group of stars.

Exercise 7.2. Another group of stars has opacity of the Kramers type $\kappa = \kappa_0 \rho T^{-3.5}$. From the equation of radiative transfer, deduce that

$$L \propto M^{5.5} R^{-0.5}$$

$$L \propto M^{5.5} R^{-0.5}.$$

The energy production rate for these stars is $\varepsilon = \varepsilon_0 \rho T^{16}$. Deduce that

$$R \propto M^{25/37}$$

$$R \propto M^{25/37}.$$

Obtain the slope of the line in the H-R diagram, which represents the positions of this group of stars.