

## Lecture 4

### Hydrostatic equilibrium

As argued in Lecture 1, the absence of changes in most stars over timescales of hours or days indicates that the forces acting on the matter in the stars are essentially perfectly balanced. Here we analyze this constraint in more detail.

#### 4.1. Equation of hydrostatic support

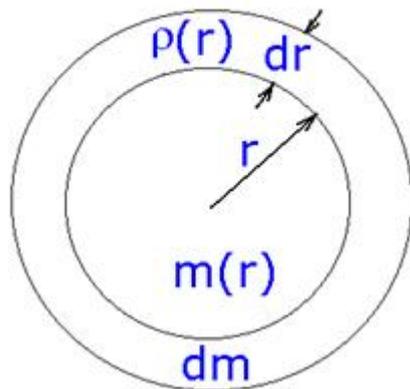


Figure 4.1. Mass shell in a spherically symmetric star. In hydrostatic equilibrium, the pressure difference over the shell balances the gravitational force on  $dm$  due to the mass  $m(r)$  interior to  $r$ .

We consider a spherical shell in the star, between the distance  $r$  and the distance  $r + dr$  from the center of the star. To obtain the equation of motion for the cell, we must evaluate the forces acting on the shell. One force is gravity. It is well known that the gravitational force at the shell arises solely from the mass contained within the shell. Let this mass be  $m$ . Then the gravitational acceleration is  $-\frac{Gm}{r^2}$ , where we count forces and accelerations positive in the direction of increasing  $r$ .

The volume of the shell is  $4\pi r^2 dr$ . Hence, if the density of matter in the shell is  $\rho$ , the mass in the shell is  $4\pi r^2 \rho dr$ , and the gravitational force is

$$F_g = -4\pi r^2 \rho \frac{Gm}{r^2} dr.$$

$$F_g = -4\pi r^2 \rho \frac{Gm}{r^2} dr. \quad (4.1)$$

The second force arises from the pressure difference across the shell. Since pressure  $P$  is defined as force per unit area, this force is

$$F_p = 4\pi r^2 P(r) - 4\pi r^2 P(r + dr) \approx -4\pi r^2 \frac{dP}{dr} dr,$$

$$F_p = 4\pi r^2 P(r) - 4\pi r^2 P(r + dr) \approx -4\pi r^2 \frac{dP}{dr} dr, \quad (4.2)$$

by making a Taylor expansion of  $P$ . Hence, by balancing mass times acceleration with the combined force, the equation of motion of the shell is obtained as

$$4\pi r^2 \rho dr \frac{\partial^2 r}{\partial t^2} = F_G + F_p,$$

$$4\pi r^2 \rho dr \frac{\partial^2 r}{\partial t^2} = F_G + F_p, \quad (4.3)$$

or

$$\rho \frac{\partial^2 r}{\partial t^2} = -\rho \frac{Gm}{r^2} - \frac{dP}{dr}.$$

$$\rho \frac{\partial^2 r}{\partial t^2} = -\rho \frac{Gm}{r^2} - \frac{dP}{dr}. \quad (4.4)$$

If we demand that the star be in equilibrium, so that the acceleration vanishes, the pressure gradient must be determined by

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}. \quad \frac{dP}{dr} = -\frac{Gm\rho}{r^2}.$$

(4.5)

This is the first of the equations of stellar structure.

This equation must be supplemented by an equation relating  $m$  to the other properties of the star. It follows immediately from the definition of  $m$ , and the fact that the mass in the shell is  $4\pi r^2 \rho dr$ , that

$$dm = 4\pi r^2 \rho dr, \quad dm = 4\pi r^2 \rho dr,$$

(4.6)

and hence

$$\frac{dm}{dr} = 4\pi r^2 \rho. \quad \frac{dm}{dr} = 4\pi r^2 \rho.$$

(4.7)

This is our second equation.

## 4.2. Estimates of stellar internal pressure and temperature

From the equation of hydrostatic support (4.5), we may obtain an estimate of the central pressure  $P_c$  of a star with mass  $M$  and radius  $R$ . We make the following approximations:

- Replace  $dP/dr$  by  $-P_c/R$ .
- Replace  $m$  by  $M$ .
- Replace  $r$  by  $R$ .
- Replace  $\rho$  by the mean density, approximated as  $M/R^3$ .

Then equation (4.5) gives

$$\frac{P_c}{R} \approx \frac{GM^2}{R^5}, \quad \frac{P_c}{R} \approx \frac{GM^2}{R^5},$$

(4.8)

or

$$P_c \approx \frac{GM^2}{R^4}, \quad P_c \approx \frac{GM^2}{R^4}.$$

(4.9)

If we assume the ideal gas law, equation (3.13), we may estimate the central temperature as

$$T_c = \frac{\mu_c m_H P_c}{k \rho_c} \approx \frac{G \mu_c m_H M}{k R},$$

$$T_c = \frac{\mu_c m_H P_c}{k \rho_c} \approx \frac{G \mu_c m_H M}{k R},$$

(4.10)

where  $\mu_c$  is the central mean molecular weight. In terms of solar values we obtain

$$P_c \approx 1.1 \times 10^{15} \left( \frac{M}{M_\odot} \right)^2 \left( \frac{R}{R_\odot} \right)^{-4} \text{ Nm}^{-2},$$

$$P_c \approx 1.1 \times 10^{15} \left( \frac{M}{M_\odot} \right)^2 \left( \frac{R}{R_\odot} \right)^{-4} \text{ Nm}^{-2},$$

(4.11)

$$T_c \approx 1.9 \times 10^7 \left( \frac{M}{M_\odot} \right) \left( \frac{R}{R_\odot} \right)^{-1} \left( \frac{\mu_c}{0.85} \right) K,$$

$$T_c \approx 1.9 \times 10^7 \left( \frac{M}{M_\odot} \right) \left( \frac{R}{R_\odot} \right)^{-1} \left( \frac{\mu_c}{0.85} \right) K, \quad (4.12)$$

where the value of  $\mu_c$  was obtained from equation (3.20), with  $X=0.35$ ,  $Z=0.02$ .

The interpretation of these results is the usual: they are obviously not accurate estimates of the central pressure or temperature of a star, but they provide an order of magnitude. After all, without any prior knowledge it would be difficult to guess whether the central pressure of the Sun was  $10^{10}$ ,  $10^{20}$ , or  $10^{30}$  N m<sup>-2</sup> ! In fact the estimates are reasonably (in the case of the temperature, fortuitously) accurate: realistic computations of solar models show that the solar central pressure is  $2.4 \times 10^{16}$  N m<sup>-2</sup> and the solar central temperature is  $1.5 \times 10^7$  K. It should be noted also that the estimate for the pressure is obtained solely on the basis of Newtonian mechanics; the estimate of the temperature in addition required a minimal amount of thermodynamics. Surely a good example of the power of basic physics to provide knowledge about the internal properties of stars.

The second aspect of the simple estimates is that they indicate how the pressure and temperature scale with the stellar mass and radius. This dependence has a wider applicability. We shall later see several examples of how this scaling can be given a more precise meaning for particular types of simplified stellar models. And even for realistic stellar models, with detailed physics, one often find that the scaling provided by the simple estimates are surprisingly accurate when the stellar parameters are varied. Thus these estimates are very helpful for the interpretation of detailed numerical results.

### 4.3. Lower limit on the central pressure

It is of some interest that a strict limit can be obtained for the central pressure of a star, with no other assumptions beyond hydrostatic equilibrium. It is obtained by manipulating equation (4.5), using also equation (4.7):

$$\begin{aligned}\frac{dP}{dr} &= -\frac{Gm\rho}{r^2} = -\frac{Gm}{4\pi r^4} 4\pi r^2 \rho = -\frac{Gm}{4\pi r^4} \frac{dm}{dr} \\ &= -\frac{d}{dr} \left( \frac{Gm^2}{8\pi r^4} \right) - \frac{Gm^2}{2\pi r^5},\end{aligned}$$

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and hence

$$\begin{aligned}\frac{d}{dr} \left( P + \frac{Gm^2}{8\pi r^4} \right) &= -\frac{Gm^2}{2\pi r^5} < 0. \\ \frac{d}{dr} \left( P + \frac{Gm^2}{8\pi r^4} \right) &= -\frac{Gm^2}{2\pi r^5} < 0.\end{aligned}\tag{4.14}$$

This shows that the quantity  $\Psi(r) = P + Gm^2/(8\pi r^4)$  is a decreasing function of  $r$ . At the center  $P = P_c$ ; also, equation (4.7) shows that  $m \propto r^3$  for small  $r$ , so that the second term in  $\Psi$  vanishes at  $r=0$ . Hence  $\Psi(0) = P_c$ . At the surface  $P$  is essentially zero. Thus, from the fact that  $\Psi$  is a decreasing function of  $r$  it follows that

$$P_c = \Psi(0) > \Psi(R) = \frac{GM^2}{8\pi R^4},$$

$$P_c = \Psi(0) > \Psi(R) = \frac{GM^2}{8\pi R^4},$$

(4.15)

and this is the desired limit.

It is remarkable that this limit is a strict mathematical result, valid for any stellar model in hydrostatic equilibrium, regardless of its other properties, such as equation of state or energy transport and production. Also, it confirms that  $GM^2/R^4$  is indeed a characteristic value for the internal pressure of stars. On the other hand, the limit is fairly weak, compared with the actual solar central pressure quoted above.

#### 4.4. The virial theorem

From the equation of hydrostatic equilibrium we can derive an equation for the energetics of a star, which is of greatest importance for understanding stellar evolution. We begin by deriving an expression for the gravitational potential energy of the star. At the distance  $r$  from the centre the gravitational potential is  $-Gm/r$ , if we choose the arbitrary constant in the potential such that it is zero infinitely far from the star. Hence the total potential energy is

$$\Omega = -\int_0^R \frac{Gm}{r} 4\pi r^2 \rho dr = -4\pi \int_0^R Gm r \rho dr.$$

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This may be rewritten, by using the equation of hydrostatic support (4.5) and integrating by parts:

$$\Omega = -4\pi \int_0^R \frac{Gm\rho}{r^2} r^3 dr = 4\pi \int_0^R \frac{dP}{dr} r^3 dr$$

$$= \left[ 4\pi P r^3 \right]_0^R - 3 \int_0^R P 4\pi r^2 dr.$$

$$\Omega = -4\pi \int_0^R \frac{Gm\rho}{r^2} r^3 dr = 4\pi \int_0^R \frac{dP}{dr} r^3 dr$$

$$= \left[ 4\pi P r^3 \right]_0^R - 3 \int_0^R P 4\pi r^2 dr. \quad (4.17)$$

Here the integrated terms vanishes, since  $P=0$  at the surface  $r=R$ . Since  $4\pi r^2 dr$  is a volume element, we have

$$\Omega = -3 \int_V P dv, \quad \Omega = -3 \int_V P dv,$$

(4.18)

where integration is performed over the volume  $V$  occupied by the star. But for the ideal gas, pressure  $P$  is related with internal energy per unit volume  $u$  as  $u=3/2P$  (equation 3.21), and we finally obtain

$$\Omega = -2U, \quad \Omega = -2U,$$

(4.19)

where  $U$  is the total thermal energy of the star. This relation is called the *virial theorem*. It follows then that the total energy of the star is

$$E = \Omega + U = -U = \frac{1}{2} \Omega;$$

$$E = \Omega + U = -U = \frac{1}{2} \Omega; \quad (4.20)$$

that the total energy is negative indicates that the star is stable: the thermal energy in the star is insufficient to cause it to explode.

The last equation allows us to understand the evolution of stars where there are no sources of nuclear energy. The tendency for such a star is to contract under gravity. In this way the gravitational potential energy becomes more negative. The same is therefore true of the total energy of the star. However, since globally there has to be energy conservation, the energy lost by the star has to go somewhere else, and hence it is radiated from the star. Specifically, the luminosity of a contracting star is given by

$$L_G = -\frac{1}{2} \frac{d\Omega}{dt} \approx -\frac{1}{2} \frac{GM^2}{R^2} \frac{dR}{dt},$$

$$L_G = -\frac{1}{2} \frac{d\Omega}{dt} \approx -\frac{1}{2} \frac{GM^2}{R^2} \frac{dR}{dt},$$

(4.21)

where in the last approximate identity we used the estimate in equation (1.5) for the total gravitational potential energy. From equation (4.20) it follows also that the thermal energy  $U$  increases; so therefore does the average temperature in the gas. Of the gravitational potential energy that is released in the contraction, half is radiated away and the other half goes to heat up the gas. This demonstrates the paradoxical property of self-gravitating systems that they have a negative specific heat: as they lose energy, they become warmer.

It follows from equation (4.21) that

$$\frac{dR}{dt} \approx -2 \frac{R}{t_{KH}}, \quad \frac{dR}{dt} \approx -2 \frac{R}{t_{KH}},$$

(4.22)

where  $t_{KH}$  is the Kelvin-Helmholtz time defined in equation (1.6). This confirms that  $t_{KH}$  is a characteristic time for the gravitational contraction of a star. It might be noted that the equality in equation (4.19) shows that  $t_{KH}$  is also a characteristic time for the radiation of the thermal energy of the star. Hence changes that involve substantial losses or gains of energy can not take place on timescales shorter than  $t_{KH}$ , at least as long as hydrostatic equilibrium is nearly maintained. Correspondingly, for changes that *do* occur on much shorter timescales the changes in energy must be very small; in other words, such changes are nearly adiabatic.

Similar effects occur in later stages of stellar evolution, as nuclear fuels are exhausted in the core of the star: again, the result is a gravitational contraction which releases energy and heats up the core, until the point is reached where further nuclear reactions set in. In this case, however, the situation may be complicated by the presence elsewhere in the star of nuclear burning shells; also, one often finds that the outer parts of the star expand (which require energy to work against gravity) as the core contracts. Thus the understanding of these evolutionary phases is less straightforward; but the virial theorem still plays a central role.

When the gas can not be regarded as ideal, or effects of ionization have to be taken into account, the simple equation (4.20) must be modified; but the general principles remain the same.

## Exercises

**Exercise 4.1.** Assuming that the Sun has a constant density throughout its interior, what is the magnitude of the gravitational field (gravitational acceleration) half way in? 

**Exercise 4.2.** Show that for a star composed of an ideal gas of uniform density, the central pressure is

$$P_c = \frac{3GM^2}{8\pi R^4} \qquad P_c = \frac{3GM^2}{8\pi R^4}$$

and the central temperature is

$$T_c = \frac{GM}{2R} \frac{\mu}{\mathcal{R}} \qquad T_c = \frac{GM}{2R} \frac{\mu}{\mathcal{R}}$$