University College London
Department of Physics and Astronomy
2246 Mathematical Methods III
Coursework M1 (2007-2008)

Solutions to be handed in on Wednesday, October 31st, 2007

1. For the 3×3 matrices

$$\underline{A} = \begin{pmatrix} 3 & 1 & -3 \\ 1 & 4 & 2 \\ -3 & 2 & 5 \end{pmatrix} \quad \text{and} \quad \underline{B} = \begin{pmatrix} 0 & 4 & 1 \\ -4 & 0 & -2 \\ -1 & 2 & 0 \end{pmatrix} ,$$

evaluate the products $\underline{C} = \underline{AB}$ and $\underline{D} = \underline{BA}$. 4 MARKS

Show, that although $\underline{C} \neq \underline{D}$, the determinants of $\underline{C}$ and $\underline{D}$ are both equal to the product of the determinants of $\underline{A}$ and $\underline{B}$. 5 MARKS

Show also that the sum of the diagonal elements of $\underline{C}$ and $\underline{D}$ are the same. 2 MARKS

2. The equations

$$x_2 = \frac{1}{\sqrt{2}}\left(x_1 - z_1\right) \qquad x_3 = \frac{1}{\sqrt{2}}\left(y_2 - z_2\right) ,$$

$$y_2 = \frac{1}{2}\left(x_1 + \sqrt{2}y_1 + z_1\right) \quad y_3 = -\frac{1}{2}\left(\sqrt{2}x_2 + y_2 + z_2\right) ,$$

$$z_2 = \frac{1}{2}\left(x_1 - \sqrt{2}y_1 + z_1\right) \quad z_3 = \frac{1}{2}\left(-\sqrt{2}x_2 + y_2 + z_2\right)$$

represents rotations in three dimensions. Use matrix techniques to express the components of $\underline{r}_3$ in terms of those of $\underline{r}_1$.
What does the resultant single transformation represent geometrically? 7 MARKS

3. The matrices $\underline{A}$, $\underline{B}$ and $\underline{D}$ are related by $\underline{D} = \underline{AB}$. Given that

$$\underline{A} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 0 \\ 0 & 5 & 1 \end{pmatrix} \quad \text{and} \quad \underline{D} = \begin{pmatrix} 7 & -1 & 0 \\ 3 & 1 & -1 \\ 3 & 9 & 5 \end{pmatrix} ,$$

evaluate $\underline{A}^{-1}$. 7 MARKS
Hence derive the value of $\underline{B}$. 3 MARKS
Turn sheet over →

4. Find the eigenvalues and normalized eigenvectors of the matrix

$$\underline{A} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} .$$

6 MARKS

Show that $\underline{A}^2 = \underline{I} + 2\underline{A}$ and hence evaluate $\underline{A}^4$ and $\underline{A}^8$. 3 MARKS

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Model Answers

1. Solution to Problem 1

$$\underline{C} = \underline{A}\underline{B} = \begin{pmatrix} 3 & 1 & -3 \\ 1 & 4 & 2 \\ -3 & 2 & 5 \end{pmatrix} \begin{pmatrix} 0 & 4 & 1 \\ -4 & 0 & -2 \\ -1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 6 & 1 \\ -18 & 8 & -7 \\ -13 & -2 & -7 \end{pmatrix}$$

$$\underline{D} = \underline{B}\underline{A} = \begin{pmatrix} 0 & 4 & 1 \\ -4 & 0 & -2 \\ -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 3 & 1 & -3 \\ 1 & 4 & 2 \\ -3 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 18 & 13 \\ -6 & -8 & 2 \\ -1 & 7 & 7 \end{pmatrix}$$

$$\begin{aligned} \det \underline{C} &= \begin{vmatrix} -1 & 6 & 1 \\ -18 & 8 & -7 \\ -13 & -2 & -7 \end{vmatrix} = +1 \begin{vmatrix} -18 & 8 \\ -13 & -2 \end{vmatrix} + 7 \begin{vmatrix} -1 & 6 \\ -13 & -2 \end{vmatrix} - 7 \begin{vmatrix} -1 & 6 \\ -18 & 8 \end{vmatrix} \\ &= (36 + 104) + 7(2 + 78) - 7(-8 + 108) = 140 + 560 - 700 = 0 \end{aligned}$$

$$\begin{aligned} \det \underline{C} &= \begin{vmatrix} 1 & 18 & 13 \\ -6 & -8 & 2 \\ -1 & 7 & 7 \end{vmatrix} = -1 \begin{vmatrix} 18 & 13 \\ -8 & 2 \end{vmatrix} - 7 \begin{vmatrix} 1 & 13 \\ -6 & 2 \end{vmatrix} + 7 \begin{vmatrix} 1 & 18 \\ -6 & -8 \end{vmatrix} \\ &= = (36 + 104) - 7(2 + 78) + 7(-8 + 108) - 140 - 560 + 700 = 0 \end{aligned}$$

$$\begin{aligned} \det \underline{A} &= \begin{vmatrix} 3 & 1 & -3 \\ 1 & 4 & 2 \\ -3 & 2 & 5 \end{vmatrix} = 3 \begin{vmatrix} 4 & 2 \\ 2 & 5 \end{vmatrix} - \begin{vmatrix} 1 & -3 \\ 2 & 5 \end{vmatrix} - 3 \begin{vmatrix} 1 & -3 \\ 4 & 2 \end{vmatrix} \\ &= 3(20 - 4) - (5 + 6) - 3(2 + 12) = 48 - 11 - 42 = -5 \end{aligned}$$

$$\begin{aligned} \det \underline{B} &= \begin{vmatrix} 0 & 4 & 1 \\ -4 & 0 & -2 \\ -1 & 2 & 0 \end{vmatrix} = -4 \begin{vmatrix} -4 & -2 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -4 & 0 \\ -1 & 2 \end{vmatrix} \\ &= -4(-2) - 8 = 8 - 8 = 0 \end{aligned}$$

$$\mathrm{tr}\underline{C} = -1 + 8 - 7 = 0$$

$$\mathrm{tr}\underline{D} = 1 - 8 + 7 = 0$$

2. Solutions to 2)
   Define

$$\underline{r}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \qquad \underline{r}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \qquad \underline{r}_3 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}$$

   Further define:

$$\underline{r}_2 = \underline{A}\,\underline{r}_1 \qquad \underline{r}_3 = \underline{B}\,\underline{r}_2 \ ,$$

   with

$$\underline{A} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}$$

   and

$$\underline{B} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} .$$

   With $\underline{r}_3 = \underline{B}\,\underline{r}_2 = \underline{B}\,\underline{A}\,\underline{r}_1 = \underline{C}\,\underline{r}_1$ we obtain

$$\underline{C} = \underline{B}\,\underline{A} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   which means

$$\begin{aligned} x_3 &= y_1 \\ y_3 &= -x_1 \\ z_3 &= z_1 \end{aligned}$$

   which is a rotation by $\pi/2$ clockwise around the z-axis.

3. Solutions to 3).
   Calculate $\underline{A}^{-1}$. Matrix of minors $\underline{M}$:

$$\underline{M} = \begin{pmatrix} \begin{vmatrix} -1 & 0 \\ 5 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ 0 & 5 \end{vmatrix} \\ \begin{vmatrix} 0 & 2 \\ -1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 5 \end{vmatrix} \\ \begin{vmatrix} 0 & 2 \\ -1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 3 & -1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -1 & 3 & 15 \\ -10 & 1 & 5 \\ 2 & -6 & -1 \end{pmatrix}$$

cofactor matrix $(\underline{C})_{ij} = (-1)^{i+j}(\underline{M})_{ij}$

$$\underline{C} = \begin{pmatrix} -1 & -3 & 15 \\ 10 & 1 & -5 \\ 2 & 6 & -1 \end{pmatrix}$$

and the adjoint $\underline{A}^{\text{adj}} = \underline{C}^T$

$$\underline{A}^{\text{adj}} = \begin{pmatrix} -1 & 10 & 2 \\ -3 & 1 & 6 \\ 15 & -5 & -1 \end{pmatrix}$$

The determinant of $\underline{A}$ can be established easily from one row or column of the minor matrix to

$$\det \underline{A} = -1 + 2(15) = 29$$

and hence

$$\underline{A}^{-1} = \frac{\underline{A}^{\text{adj}}}{|\underline{A}|} = \frac{1}{29}\begin{pmatrix} -1 & 10 & 2 \\ -3 & 1 & 6 \\ 15 & -5 & -1 \end{pmatrix}$$

Since $\underline{D} = \underline{A}\,\underline{B} \Rightarrow \underline{A}^{-1}\underline{D} = \underline{B}$ we obtain

$$\underline{B} = \frac{1}{29}\begin{pmatrix} -1 & 10 & 2 \\ -3 & 1 & 6 \\ 15 & -5 & -1 \end{pmatrix}\begin{pmatrix} 7 & -1 & 0 \\ 3 & 1 & -1 \\ 3 & 9 & 5 \end{pmatrix}$$

$$= \frac{1}{29}\begin{pmatrix} 29 & 29 & 0 \\ 0 & 58 & 29 \\ 87 & -29 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 3 & -1 & 0 \end{pmatrix}$$

4. Solution to 4.)

$$\underline{A} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

hence the characteristic equation is given by

$$\det(\underline{A} - \lambda\underline{I}) = \begin{vmatrix} -\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\begin{aligned} & -\lambda(2-\lambda) - 1 && = 0 \\ \Leftrightarrow \quad & \lambda^2 - 2\lambda - 1 && = 0 \\ \Leftrightarrow \quad & \lambda^2 - 2\lambda + 1 - 2 && = 0 \\ \Leftrightarrow \quad & (\lambda - 1)^2 && = 2 \\ \lambda_{1,2} \quad & = && 1 \pm \sqrt{2} \end{aligned}$$

And hence the Eigenvector to $\lambda_1 = 1 - \sqrt{2}$:

$$\begin{pmatrix} -1+\sqrt{2} & 1 \\ 1 & 2-1+\sqrt{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and we obtain

$$(-1+\sqrt{2})x_1 + x_2 = 0$$

hence

$$x_2 = (1-\sqrt{2})x_1$$

and as Eigenvector

$$\underline{v}_1 = \begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix}$$

and finally the normalized Eigenvector

$$\hat{\underline{v}}_1 = \frac{1}{\sqrt{4-2\sqrt{2}}} \begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix}$$

The Eigenvector to $\lambda_2 = 1 + \sqrt{2}$:

$$\begin{pmatrix} -1-\sqrt{2} & 1 \\ 1 & 1-\sqrt{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and we obtain

$$(-1-\sqrt{2})x_1 + x_2 = 0$$

hence

$$x_2 = (1+\sqrt{2})x_1$$

and as Eigenvector

$$\underline{v}_2 = \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix}$$

and the normalized Eigenvector

$$\hat{\underline{v}}_2 = \frac{1}{\sqrt{4+2\sqrt{2}}} \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix}$$

The matrix $\underline{A}^2$ is given by

$$\underline{A}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$$

and

$$\underline{I} + 2\underline{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$$

$$\Rightarrow \underline{A}^2 = \underline{I} + 2\underline{A}$$

and then:

$$\underline{A}^4 = \left(\underline{A}^2\right)^2 = (\underline{I} + 2\underline{A})^2 = \underline{I}^2 + 4\underline{A}^2 + 4\underline{A}$$

where we have used for the third equality that the identity matrix $\underline{I}$ commutes with all matrices. We obtain then

$$= \underline{I} + 4(\underline{I} + 2\underline{A}) + 4\underline{A} = 5\underline{I} + 12\underline{A} = \begin{pmatrix} 5 & 12 \\ 12 & 29 \end{pmatrix}$$

also

$$\underline{A}^9 = \left(\underline{A}^4\right)^2 = (5\underline{I} + 12\underline{A})^2 = 25\underline{I} + 144\underline{A}^2 + 120\underline{A}$$

$$= 25\underline{I} + 120\underline{A} + 144(\underline{I} + 2\underline{A}) = 169\underline{I} + 408\underline{A} = \begin{pmatrix} 169 & 408 \\ 408 & 985 \end{pmatrix}$$