Topic 22 — Interference of Several Sources *FGT1051-1053, AF914-917*

sum of signals from several dipoles *AF914*

If we have a large number of sources, we can add them together using the phasor or the complex number picture. Here we use the complex number aproach.

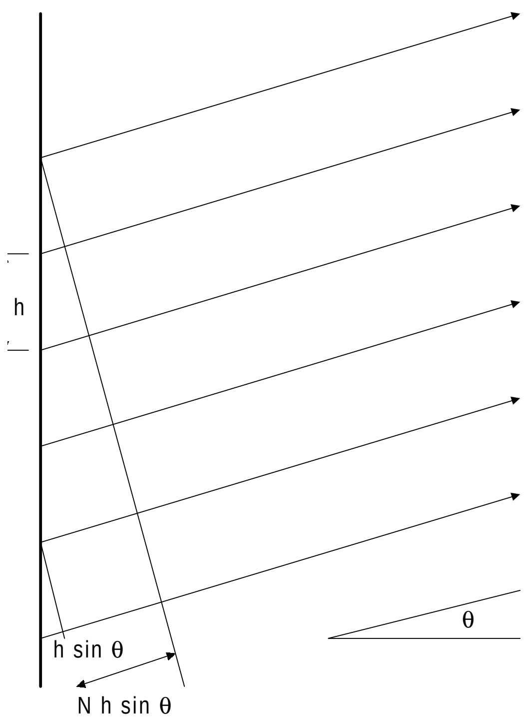

Figure L22.1: The geometry of several interfering sources.

Suppose we are a long distance r from such an array, looking at an angle θ off axis (see figure L22.1). If the sources are h apart, then the phase difference δ between successive sources will be

$$\delta = \frac{2\pi}{\lambda} h \sin\theta$$

and if the signal from one slit is

$$E_1(\omega, t, r, \theta)$$

then the total signal from N of them will be

$$E(\omega, t, r, \theta) = E_1(\omega, t, r, \theta) + E_1(\omega, t, r, \theta)e^{i\delta} + E_1(\omega, t, r, \theta)e^{2i\delta}...$$

which is a geometric series with common ratio $e^{i\delta}$, so

$$E(\omega, t, r, \theta) = E_1(\omega, t, r, \theta)\frac{1 - e^{Ni\delta}}{1 - e^{i\delta}}$$

which can be simplified to

$$E(\omega, t, r, \theta) = E_1(\omega, t, r, \theta)e^{i(N-1)\delta/2}\frac{\sin(N\delta/2)}{\sin(\delta/2)}.$$

How does this function vary with N? When δ is zero or any other integer multiple of 2π amplitude has the form $0/0$, so we have to resort to l'Hôpital's rule do find the limit, which is N. Thus whenever

$$\sin\theta = m\frac{\lambda}{h}$$

where m is any integer, there is a peak of amplitude N, because

$$\lim_{\delta to 2m\pi}\left(\frac{\sin(N\delta/2)}{\sin(\delta/2)}\right) = N.$$

Of course, we might have realised this from thinking of the phasor diagram: the best we can do is to add all the signals in phase, and then the amplitude will just be N times the one-slit amplitude.

Next, we note that if the sources are all ideal line sources, they will radiate uniformly in all directions, so that E_1 is independent of θ.

We get the intensity by squaring the modulus of the amplitude and so

$$I(\theta) = I(0)\left[\frac{\sin\left(\frac{N\pi h}{\lambda}\sin(\theta)\right)}{N\sin\left(\frac{\pi h}{\lambda}\sin(\theta)\right)}\right]^2.$$

We can see that whenever whenever $\frac{N\pi h}{\lambda}\sin(\theta)$ is a multiple of π we have a zero, unless $\frac{\pi h}{\lambda}\sin(\theta)$ is a multiple of π, in which case we have a maximum of height $I(0)$. Thus the widths of the maxima decrease as N increases – see figures L22.2 and L22.3. The number of lines in these figures are much smaller than would be found in a realistic grating, but they do show the point that the principal maxima decrease in width as the number of lines increases. One can also count the subsidiary maxima, and verify that there are just $N-2$ such subsidiary peaks between principal maxima.

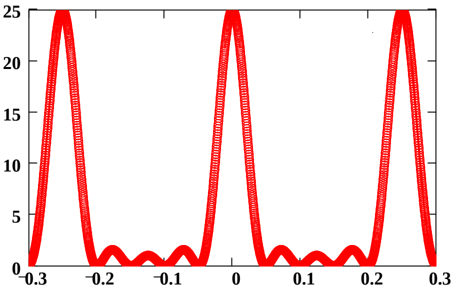

Figure L22.2: The interference pattern produced by five line sources, showing intensity plotted as a function of angle in radians.

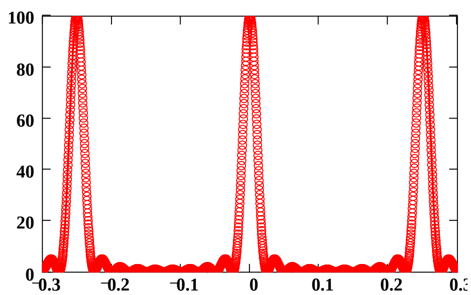

Figure L22.3: The interference pattern produced by ten line sources, showing intensity plotted as a function of angle in radians.

This, of course, is the principle of the diffraction grating.

Note also that if we have only two slits, $N=2$, we have

$$I(\theta) = I(0)\left[\frac{\sin\left(\frac{2\pi h}{\lambda}\sin(\theta)\right)}{2\sin\left(\frac{\pi h}{\lambda}\sin(\theta)\right)}\right]^2$$

$$= \quad I(0)\left[\frac{2\sin\left(\frac{\pi h}{\lambda}\sin(\theta)\right)\cos\left(\frac{\pi h}{\lambda}\sin(\theta)\right)}{2\sin\left(\frac{\pi h}{\lambda}\sin(\theta)\right)}\right]^2$$

$$= \quad I(0)\left[\cos\left(\frac{\pi h}{\lambda}\sin(\theta)\right)\right]^2$$

and if the angle θ is small we may write

$$\sin(\theta) \approx \tan(\theta) = y/x$$

to recover the previous expression for Young's slits

$$I(\theta) = I(0)\left[\cos\left(\frac{\pi hy}{\lambda x}\right)\right]^2.$$

laboratory example

Note that there is an experiment in the undergraduate laboratory which uses the half-millimetre marks on a metre rule as a reflection diffraction grating, with an infrared laser as the source. A warning: the half-millimetre marks are slightly shorter than the millimetre marks, so if the system is aligned incorrectly one uses the millimetre marks instead as the grating, but assumes them to be half a millimetre apart, leading to a wavelength which is twice what it should be.

phasor diagrams *FGT915-916, AF915*

The results of this section can also be obtained by methods based on phasor diagrams. In fact, this may help to clarify what the factors of N are all about in the intensity pattern. In the straight-through direction all the signals add in phase, so the total amplitude is N times the amplitude from one slit, or the total intensity is N^2 times the intensity from one slit. So if we refer all our intensities to this straight-through intensity, we want the function by which it is multiplied to have a value of 1 at $\theta = 0$. That is why we incorporate the N in the denominator of the expression in square brackets.

fading of edges of fringe patterns *AF912*

As with the two-slit situation, the longer path differences which occur at large angles require long coherence times, and thus the edges of diffraction patterns (high angles) tend to be less distinct than the centres (small angles).

L22.1 Resolving power of diffraction grating

We now know the angular positions of principal maxima of a diffraction grating, and we have seen that the widths of these maxima depend on the number of lines in the grating. By putting these together we can draw conclusions about the ability of the grating to distinguish between spectral lines of similar wavelength, such as the two yellow lines of sodium. In order to do this, though, we need a way of deciding whether two closely-spaced lines will appear to the eye to be separate.

Rayleigh criterion

Rayleigh's criterion states that two similar diffraction patterns can just be separated if the first zero of one pattern falls on the central peak of the other.

As shown in figure L22.4, this gives an adequate dip in intensity. This is Rayleigh's criterion for resolution. The rule is somewhat arbitrary, but widely accepted.

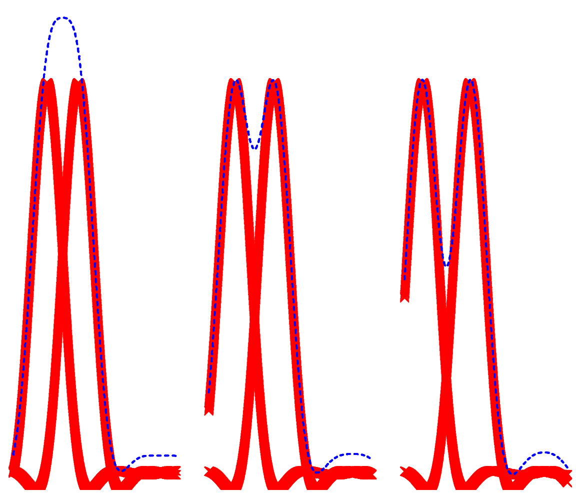

Figure L22.4: Rayleigh's criterion for two lines to be distinguishable, illustrated with pairs of lines separated by 0.8, 1.0 and 1.2 times the Rayleigh criterion.

Resolution and the grating

We can apply the Rayleigh criterion to the ability of a diffraction grating to separate different wavelengths. We know that the position of a principal maximum is given by

$$\frac{\pi h}{\lambda}\sin(\theta_{\mathrm{max}}) = m\pi,$$

or

$$\sin(\theta_{\mathrm{max}}) = m\frac{\lambda}{h} \tag{L22.1}$$

where m is an integer.

If we write the maximum position as

$$N\frac{\pi h}{\lambda}\sin(\theta_{\mathrm{max}}) = Nm\pi,$$

it is clear that the position of the first zero to one side of this principal maximum of a diffraction pattern is given by

$$N\frac{\pi h}{\lambda}\sin(\theta_{\mathrm{min}}) = (Nm+1)\pi$$

or,

$$\sin(\theta_{\mathrm{min}}) - \sin(\theta_{\mathrm{max}}) = \frac{\lambda}{h}\left(m + \frac{1}{N}\right) - m\frac{\lambda}{h} = \frac{\lambda}{h}\,\frac{1}{N}.$$

For small angles, the angular separation of the peak and the minimum is

$$\theta_{\mathrm{min}} - \theta_{\mathrm{max}} \approx \frac{\lambda}{h}\,\frac{1}{N}.$$

Note that this expression gives us the half-width of the peak, and so it is the quantitative expression of our observation that the peak gets narrower as the number of lines in the grating, N, increases.

Now the angular separation between the mth order principal maxima from two wavelengths which differ by $\Delta\lambda$ may be obtained by differentiating equation L22.1:

$$\cos(\theta)\mathrm{d}\theta = m\frac{\mathrm{d}\lambda}{h}$$

or, for small angles,

$$\Delta\theta \approx m\frac{\Delta\lambda}{h}$$

. We can distinguish the peaks provided the peak of one falls on or beyond the first minimum of the other, i.e.

$$\frac{\lambda}{h}\frac{1}{N} < m\frac{\Delta\lambda}{h}$$

or

$$\frac{\lambda}{\Delta\lambda} < mN$$

.

The resolving power is defined as

$$\frac{\lambda}{(\Delta\lambda)_{\min}} = mN,$$

thus the resolving power depends on the order of diffraction and on the number of lines in the grating.

We normally choose to work at the largest order possible. Note that if we go to high order we need to treat the angles rather more carefully (we are beyond the small angle approximation).

For example, a grating 10 mm wide and ruled with 100 lines per mm will contain a total of 1000 lines, so in the second order it will have a resolving power of 2000. This would allow it to resolve a wavelength difference of 0.3 nm in the region of 590 nm, that is, it could just resolve the sodium lines at 589.0, 589.6 nm.

Note that there is a limit to the maximum order one can observe. A principal maximum occurs when

$$\frac{\pi h}{\lambda}\sin(\theta) = m\pi$$

and as the largest value of $\sin(\theta)$ is 1, we have

$$m \le \frac{h}{\lambda}.$$

One might ask, then, why one should not simply increase the order of the diffraction, m, by increasing the slit spacing. There are two reasons

1. we will eventually have overlap between different orders, when

$$h\sin(\theta) = (m+1)\lambda = m(\lambda + \Delta\lambda),$$

which defines a *free spectral range*

$$\Delta\lambda = \frac{\lambda}{m}.$$

2. the longitudinal correlation will be too small for effective interference with the longer path-length differences.

L22.2 Other related cases

In astronomical radio telescopes, one has effectively a diffraction grating in reverse: each aerial plays the role of one of the slits in the diffraction grating, and the phase differences between signals arriving from distant sources give rise to maxima and minima of signal.

Reflection gratings behave in a similar manner: the expressions derived for the transmission grating above (which assumed light incident normally) would be equally applicable to a reflection grating with light incident normally. Reflection diffraction gratings can be seen with, for example, compact discs.