Topic 16 — Impedance Matching: II

T16.1 Quarter-wave plates – alternative approach

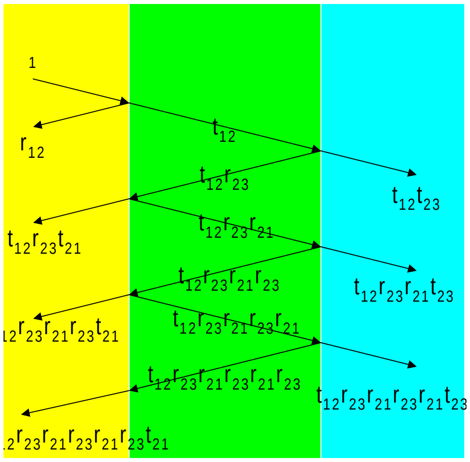

Figure T16.1: Multiple reflections in a coating. Note that only the amplitudes are shown here: when the separate reflections or transmissions are added to give the totals, appropriate phase factors must be included to account for the distance travelled in the layer.

There is another way of solving the quarter-wave matching problem, which results in the same mathematical expressions but is perhaps physically more appealing. Let's look at all the reflections and transmissions that occur in a three-layer system, as in figure T16.1. To condense the notation slightly, let us write the reflection coefficient for a wave incident from medium 1 onto medium 2 as r_{12}:

$$\mathtt{r}_{12} = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

and the coefficient for a wave going the other way

$$\mathtt{r}_{21} = \frac{Z_2 - Z_1}{Z_1 + Z_2} = -\mathtt{r}_{12}.$$

Then (see diagram) we can write the overall reflection coefficient as the sum of all the separate contributions, which have made no, one, two, etc. return trips through the layer:

$$\mathtt{r} = \mathtt{r}_{12} + \mathtt{t}_{12}\mathtt{r}_{23}e^{i\phi}\mathtt{t}_{21} + \mathtt{t}_{12}\mathtt{r}_{23}e^{i\phi}r_{21}\mathtt{r}_{23}e^{i\phi}\mathtt{t}_{21} + \ldots$$

where the phase factor accounts for the change of phase as the wave travels through the central region of thickness l in which the wavevector is k_2

$$e^{i\phi} = e^{-2ik_2 l}.$$

Then $\mathtt{r}$ may be rewritten as a first term plus a geometric series with common ratio

$$\mathtt{r}_{21} r_{23} e^{i\phi}$$

as

$$\mathtt{r} = \mathtt{r}_{12} + \mathtt{t}_{12}\mathtt{r}_{23}\mathtt{t}_{21} e^{i\phi} \frac{1}{1 - \mathtt{r}_{21}\mathtt{r}_{23} e^{i\phi}}$$

or

$$\mathtt{r} = \frac{\mathtt{r}_{12} + (\mathtt{t}_{12}\mathtt{r}_{23}\mathtt{t}_{21} - \mathtt{r}_{12}\mathtt{r}_{21}\mathtt{r}_{23})\mathtt{e}^{\mathtt{i}\phi}}{1 - \mathtt{r}_{21}\mathtt{r}_{23}\mathtt{e}^{\mathtt{i}\phi}}$$

We can simplify this if we use an important general result:

$$\mathtt{t}_{12}\mathtt{t}_{21} - \mathtt{r}_{12}\mathtt{r}_{21} = \frac{4Z_1 Z_2 - (Z_1 - Z_2)(Z_2 - Z_1)}{(Z_1 + Z_2)^2} = 1.$$

Thus

$$\mathtt{r} = \frac{\mathtt{r}_{12} + \mathtt{r}_{23} e^{i\phi}}{1 - \mathtt{r}_{21}\mathtt{r}_{23} e^{i\phi}}. \qquad \text{(T16.1)}$$

Again, if we want $\mathtt{r} = 0$ we must have both the real and imaginary parts equal to zero. To obtain a real denominator, we multiply the top and bottom of equation T16.1 by the complex conjugate of the denominator,

$$1 - \mathtt{r}_{21}\mathtt{r}_{23} e^{-i\phi}$$

to get

$$\mathtt{r} = \frac{\mathtt{r}_{12} - \mathtt{r}_{23}\mathtt{r}_{21}\mathtt{r}_{23} + \mathtt{r}_{23} e^{i\phi} - \mathtt{r}_{12}\mathtt{r}_{21}\mathtt{r}_{23} e^{-i\phi}}{1 - 2\mathtt{r}_{21}\mathtt{r}_{23}\cos(\phi) + \mathtt{r}_{21}^2\mathtt{r}_{23}^2}.$$

This will be zero if both the real and imaginary parts of the numerator are zero. Using de Moivre's theorem to write

$$e^{\pm i\phi} = \cos(\phi) \pm i\sin(\phi),$$

the imaginary part is

$$\mathtt{r}_{23}(1 + \mathtt{r}_{12}\mathtt{r}_{21})\sin(\phi) = 0$$

or

$$\mathtt{r}_{23}(1-\mathtt{r}_{12}^2)\sin(\phi)=0$$

But we know that neither of the terms in rs is zero because if all the media are different all of the rs are non-zero but less than one in magnitude, so that $\phi=\pi$ (or an integer multiple of π), $l=\lambda_2/4$.

If $\phi=\pi$, $\cos(\phi)=-1$, so the real part gives

$$\mathtt{r}_{12}-\mathtt{r}_{23}\mathtt{r}_{21}\mathtt{r}_{23}-\mathtt{r}_{23}+\mathtt{r}_{12}\mathtt{r}_{21}\mathtt{r}_{23}=0$$

because $\phi=\pi$ so $\cos(\phi)=-1$, or

$$\mathtt{r}_{12}(1-\mathtt{r}_{12}\mathtt{r}_{23})-\mathtt{r}_{23}(1-\mathtt{r}_{12}\mathtt{r}_{23})=0.$$

Therefore

$$\mathtt{r}_{12}=\mathtt{r}_{23},$$

(again, the possibility $\mathtt{r}_{12}=1/\mathtt{r}_{23}$ being ruled out because both the rs must have magnitude less than unity if media 1 and 2 are different).

Writing this in terms of impedances,

$$\begin{aligned}(Z_1-Z_2)(Z_2+Z_3) &= (Z_2-Z_3)(Z_1+Z_2)\\ Z_1Z_2-Z_2^2-Z_2Z_3+Z_1Z_3 &= Z_1Z_2-Z_1Z_3-Z_2Z_3+Z_2^2\\ Z_2^2 &= Z_1Z_3\end{aligned}$$

as before.

The point here is that there are always two ways of tackling problems of this kind,

- Start with total incident, reflected and transmitted waves and satisfy boundary conditions;
- Look at each reflection and transmission and sum

and they are equivalent because of the linearity of the wave equation.

Note that this is an example of an *interference phenomenon*: we have set up the amplitudes and phases so that when we add up all the reflected signals they cancel exactly.

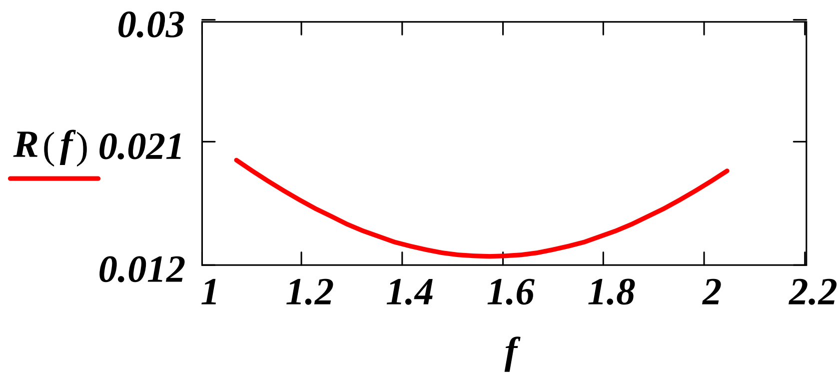

Figure T16.2: The reflected power from an air/glass interface coated with a quarter-wave layer of magnesium fluoride. The parameter f is $4\pi l/\lambda$, where l is the coating thickness and λ is the wavelength. The factor of 2 in wavelength shown would be enough to span the visible spectrum, with reflectivity optimised in the green.

Coated glass lenses

An ideal matching layer would have a refractive index $n_{\mathrm{match}} = \sqrt{n_{\mathrm{air}} n_{\mathrm{glass}}} = 1.23$. For practical reasons, involving getting a material which will stick well to glass, will match the thermal expansivity of the glass, and is relatively hard, Magnesium Fluoride ($n_{\mathrm{MgF_2}} = 1.38$) is often used as a coating material, and as figure T16.2 shows this gives a significant decrease in reflectivity, from about 4 percent for the uncoated lens to about 1 percent in the centre of the visible regime for the coated lens. In drawing the figure we have used the expression for the reflected intensity (which you need not derive)

$$R(\phi) = \frac{\left(\frac{Z_1}{Z_3}+1\right)^2 \cos^2(\phi) + \left(\frac{Z_1}{Z_2}+\frac{|_2}{Z_3}\right)^2 \sin^2(\phi) - 4\left(\frac{Z_1}{Z_3}\right)}{\left(\frac{Z_1}{Z_3}+1\right)^2 \cos^2(\phi) + \left(\frac{Z_1}{Z_2}+\frac{|_2}{Z_3}\right)^2 \sin^2(\phi)}$$

As figure T16.3 shows, this represents a significant improvement for the multilens system.

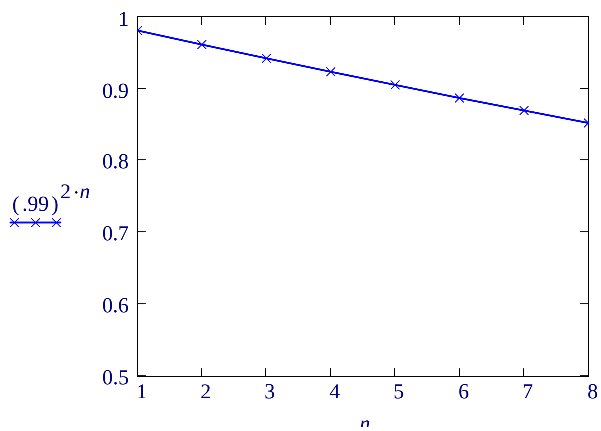

Figure T16.3: Decrease of transmitted energy as light propagates through a series of (air/glass/air) interfaces, assuming 99 percent transmission at each interface.