Topic 15 — Impedance Matching

T15.1 Matching air and glass

We saw in the last lecture that *any* interface at which the impedance changes will result in a reflected wave, and therefore (remember that we worked out both *amplitude* and *energy* reflection and transmission coefficients) in a decrease in transmitted energy. We looked at a mechanical example, but the same holds for light, where it is significant in optical instruments where the usual interface is one between air and glass. For air, with refractive index $n_{\rm air} = 1$, and glass ($n_{\rm glass} = 1.52$) the reflection coefficient for light incident from air (for, as mentioned before, the electric field) is

$$\mathtt{r} = \frac{n_{\rm air} - n_{\rm glass}}{n_{\rm air} + n_{\rm glass}} = \frac{1 - 1.52}{1 + 1.52} = -0.21,$$

corresponding to an energy reflection coefficient $R = 4$ percent. This may not appear to be much for a single lens, but build up a complex optical instrument with many lenses and the reflections can be quite significant, as shown in figure T15.1.

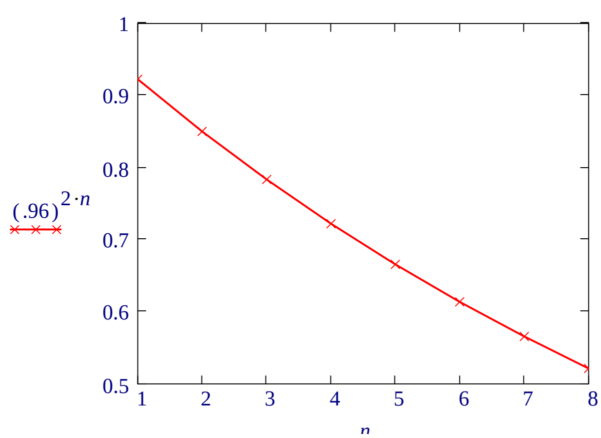

Figure T15.1: Decrease of transmitted energy as light propagates through a series of (air/glass/air) interfaces, assuming 96 percent transmission at each interface.

T15.2 Impedance matching - quarter-wave plates *P114-P117*

What the expression for r tells us

$$\mathtt{r} = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

is that *any* discontinuity will give a reflected wave. This gives a problem in all sorts of situations. In a microwave communication system, for example, we do not want to have power reflected back from the antenna to the generator. In optical systems, we do not want to have reflections from lenses which might give multiple images.

We can kill reflections with a matching system, introducing an intermediate segment in the system. Let's work this system through for a general case[1], in which we have joins at $x = 0$ and $x = l$ between media with impedances Z_1, Z_2 and Z_3. Then in region 1 let

$$\xi_1(x,t) = e^{i(\omega t - k_1 x)} + \mathtt{r}e^{i(\omega t + k_1 x)},$$

in region 2

$$\xi_2(x,t) = ae^{i(\omega t - k_2 x)} + be^{i(\omega t + k_2 x)},$$

and in region 3

$$\xi_3(x,t) = \mathtt{t}e^{i(\omega t - k_3 x)}.$$

The usual matching conditions then give at the (1-2) interface

$$1 + \mathtt{r} = a + b \tag{T15.1}$$

and

$$Z_1 - Z_1\mathtt{r} = aZ_2 - bZ_2 \tag{T15.2}$$

whilst at the (2-3) interface

$$ae^{-ik_2 l} + be^{ik_2 l} = \mathtt{t}e^{-ik_3 l} \tag{T15.3}$$

and

$$aZ_2e^{-ik_2 l} - bZ_2e^{ik_2 l} = \mathtt{t}Z_3e^{-ik_3 l}. \tag{T15.4}$$

[1]Be careful if you follow this through in Pain's book: Pain's notation is different from mine, and in particular his coefficients r are *not* reflection coefficients, but ratios of impedances.

Eliminate t by multiplying equation T15.3 by Z_3 and subtracting equation T15.4 from it:

$$a(Z_3 - Z_2)e^{-ik_2 l} + b(Z_3 + Z_2)e^{ik_2 l} = 0. \tag{T15.5}$$

Under what circumstances can we make $\mathtt{r}$ zero? Since this would require both

$$a + b = 1$$

from equation T15.1 and

$$a - b = Z_1/Z_2$$

from equation T15.2, so we must have

$$a = 1/2 + Z_1/2Z_2 \tag{T15.6}$$

and

$$b = 1/2 - Z_1/2Z_2. \tag{T15.7}$$

This requires, substituting equations T15.6 and T15.7 back into equation T15.5,

$$(1/2 + Z_1/2Z_2)(Z_3 - Z_2)e^{-ik_2 l} + (1/2 - Z_1/2Z_2)(Z_3 + Z_2)e^{ik_2 l} = 0$$

or

$$Z_3 \cos(k_2 l) + iZ_2 \sin(k_2 l) - i(Z_1/Z_2)Z_3 \sin(k_2 l) - (Z_1/Z_2)Z_2 \cos(k_2 l) = 0,$$

giving

$$(Z_3 - Z_1)\cos(k_2 l) + i[(Z_2^2 - Z_1 Z_3)/Z_2]\sin(k_2 l) = 0. \tag{T15.8}$$

Now note from the definition of the problem that the two media we are trying to match are different, so that Z_1 differs from Z_3. In order to make both the real and complex parts of the expression in equation T15.8 zero we need $\cos(k_2 l) = 0$ to make the real part zero, which we can do if $l = \lambda_2/4$ (or, indeed, any odd number of quarter-wavelengths). But then $\sin(k_2 l)$ will not be zero, so for the imaginary part to be zero we must have $Z_2^2 = Z_1 Z_3$. That is

> Perfect transmission between two media, with no reflection, may be achieved by inserting a matching layer with an impedance which is the geometric mean of the impedances of the two original media, and with a thickness which is one quarter of a wavelength in the matching medium.

This is called *quarter-wave matching*, and is used in areas ranging from electronic engineering to coating of spectacle lenses. Note that it only works properly for one frequency, so there will be compromises - your coated camera or binocular lenses are not perfectly non-reflecting, but tend to have a purplish tinge when viewed in reflection. This is because the non-reflecting coating is designed to work best in the green region of the spectrum, so less green is reflected than red and blue. Correspondingly, the transmitted light, and what you see through the lens, is tinted slightly green. The basic reason why the quarter-wave system works is that the choice of thickness ensures that the first reflection from the front of the matching layer is π out of phase with the first reflection from the interface between the matching layer and the second material. This is an example of *interference.*