Topic 11 - Acoustic Waves in Fluids and Impedance

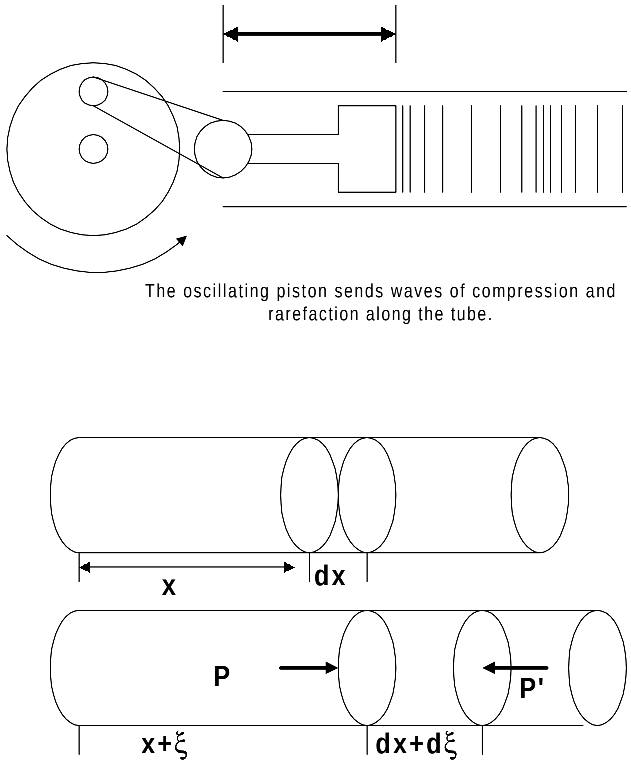

Figure T11.1: The propagation of a wave in a fluid.

T11.1 Bulk Modulus *FGT390-392, AF758-760*

Imagine a tube of fluid being driven by a sinusoidally moving piston. This will produce regions of compression and rarefaction. If the ambient pressure is P_0, the resulting pressure may be written

$$P = P_0 + p$$

where p is the change induced by the piston. If we take a fixed mass of fluid, which had a volume V_0 under the pressure P_0, it is not going to be moved bodily all the way along the pipe, but it is going to have its volume altered, say to

$$V = V_0 + v.$$

Now the degree to which a fluid can be compressed is measured by its bulk modulus:

$$B = -\frac{\text{change in pressure}}{\text{fractional change in volume}} = -\frac{\mathrm{d}P}{\mathrm{d}V/V} = -V\frac{\mathrm{d}P}{\mathrm{d}V}. \tag{T11.1}$$

Note the sign: increasing pressure results in decreased volume, so the negative sign ensures that the bulk modulus comes out positive.

T11.2 Wave in a fluid

Now we derive the equation of motion for an element of the fluid. The derivation follows very closely that for the rod. Consider (figure T11.1) an element of the tube of cross-sectional area A and initial thickness $\mathrm{d}x$. As a result of the disturbance there is a change in thickness $\mathrm{d}\xi$

$$\mathrm{d}\xi = \frac{\partial \xi}{\partial x}\mathrm{d}x,$$

which corresponds to a fractional increase in volume

$$\frac{A\mathrm{d}\xi}{A\mathrm{d}x} = \frac{\partial \xi}{\partial x}$$

the *volume strain*, and therefore to a change in pressure

$$p = -B_{\mathrm{a}}\frac{\partial \xi}{\partial x}.$$

But the difference in pressure across the element of fluid is

$$P' - P = P(x + \mathrm{d}x) - P(x) = \left(P(x) + \frac{\partial P}{\partial x}\mathrm{d}x\right) - P(x) = \frac{\partial P}{\partial x}\mathrm{d}x$$

but, as it is only the disturbing pressure p which varies with x,

$$P' - P = \frac{\partial p}{\partial x}\mathrm{d}x.$$

The force on the element is $(P' - P)A$ in the negative x direction, the mass of the element is $\rho_0 A\mathrm{d}x$ (the density will be slightly altered, but we may ignore the product of the change in density and the change in pressure as it is small compared with the terms we keep), so

$$-\frac{\partial p}{\partial x} A\mathrm{d}x = \rho_0 A\mathrm{d}x \frac{\partial^2 \xi}{\partial t^2}$$

and substituting for the pressure we get

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{B_\mathrm{a}}{\rho_0}\frac{\partial^2 \xi}{\partial x^2} \qquad \text{(T11.2)}$$

that is, a wave with velocity $v = \sqrt{B_\mathrm{a}/\rho_0}$.

Note that so far nothing we have said has been specific to any type of fluid — we could use this formula to describe sound waves in, for example, a liquid such as water or a gas such as air.

T11.3 Sound waves in a gas

Let us now look specifically at a gas. What is B? That depends on the conditions. If the pressure is changed quickly, but the changes in pressure are not too great, so that the local heating and cooling which would accompany compression and expansion are not smoothed out by thermal conduction, the process is adiabatic. This is the usual condition for sound in a gas, so

$$PV^\gamma = \text{constant} \qquad \text{(T11.3)}$$

where γ is a constant characteristic of the type of gas.

Then, differentiating

$$V^\gamma \mathrm{d}P + \gamma P V^{\gamma-1}\mathrm{d}V = 0$$

Thus

$$-V\frac{\mathrm{d}P}{\mathrm{d}V} = \gamma P = B_\mathrm{a}$$

where 'a' is for 'adiabatic'. For normal sound waves the pressure changes are *tiny* (about 10^{-10} of an atmosphere for a sound wave which is just audible at 1 kHz) so the conditions are met.[1]

[1]

$$\gamma = \frac{\partial S/\partial T)_p}{\partial S/\partial T)_V} = \frac{\partial S/\partial V)_p \partial p/\partial S)_V}{\partial T/\partial V)_p \partial p/\partial T)_V} = \frac{\partial p/\partial V)_S}{\partial p/\partial V)_T}$$

Thus the wave velocity $v = \sqrt{B_\mathrm{a}/\rho_0} = \sqrt{\gamma P_0/\rho_0}$.

For a typical diatomic gas at NTP we have $\rho_0 \approx 1$ kg m^{-3}, $\gamma = 7/5$, $P_0 \approx 10^5$ Pa, so $v \approx \sqrt{1.4 \times 10^5} \approx 374$ m s^{-1}. The value usually given for air is 330 m s^{-1}.

Characteristic impedance

T11.1 general form

Impedance describes the 'response' of a material carrying waves to a 'force'. I have used inverted commas round 'force' and 'response' as these words are being used in a general sense, and the ideas apply to electrical as well as to mechanical systems. Table T11.1 gives some examples.

Note that in all cases the product of the 'force' and the 'response' gives an energy flux: for example the electrical power IV, or a pressure (force/area) multiplied by a velocity giving a rate of doing work per unit area.

As a detailed example, consider pressure waves in a gas. The most useful definition of the 'force' is the pressure. The product of a pressure and a velocity will give a power per unit area, and this gives the useful definition

$$Z = \text{Specific acoustic impedance} = \frac{\text{excess pressure}}{\text{particle velocity}} = \frac{p}{\dot{\xi}}. \qquad \text{(T11.4)}$$

But we know from the equations we had before that

$$p = -B_\mathrm{a}\frac{\partial \xi}{\partial x}$$

and if we have a wave of the form

$$\xi = \xi_0 e^{i(\omega t - kx)}$$

we get

$$Z = \frac{B_\mathrm{a} i k \xi_0 e^{i(\omega t - kx)}}{i\omega \xi_0 e^{i(\omega t - kx)}} = \frac{B_\mathrm{a} k}{\omega} = \frac{B_\mathrm{a}}{c} = \rho_0 c = \sqrt{B_\mathrm{a}\rho_0}.$$

This is a constant, which is independent of the frequency of the wave: it is a characteristic of the material itself.

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and C_p differs from C_V by the term $p\partial V/\partial T)_p$, which is simply the gas constant R for a perfect gas.

Wave system	'Force'	'Response'	Impedance
Transverse wave on string	Force F	Velocity v	$F/v = \rho c = \sqrt{T\mu}$
Pressure wave in fluid	Excess pressure p	Velocity $\dot{\xi}$	$p/\dot{\xi} = \rho c = \sqrt{B\rho}$
Elastic wave on rod	Stress σ	Velocity $\dot{\xi}$	$\sigma/\dot{\xi} = \rho c = \sqrt{Y\rho}$
Electric current	Voltage V	Current I	$V/I = R$
Electromagnetic wave	Electric field E	Magnetic field H	$E/H = \sqrt{\mu_r\mu_0/\epsilon_r\epsilon_0}$
	Usually $\mu_r = 1$, and so $Z = Z_0/\sqrt{\epsilon_r} = Z_0/n$		
	T: tension; μ: mass per unit length ρ: density; c wave speed; B: bulk modulus σ: stress; Y: Young's modulus R: electrical resistance μ_r: relative permeability ϵ_r: relative permittivity (dielectric constant) μ_0: permeability of free space; ϵ_0: permittivity of free space Z_0: impedance of free space; n: refractive index		

Table T11.1: Impedances for several wave systems.