Topic 10 - Acoustic Waves

Let us now move away from the problem of transverse waves on a string to talk about waves in solids and in gases. First, consider an elastic rod. Here the displacements we will consider are along the rod - *longitudinal polarisation* - and correspond to compressing and expanding the rod.

T10.1 Elastic waves in a rod *FGT387, AF754-757*

Here we consider *longitudinal* waves, that is waves in which the displacement of the material is in the same direction as the direction of travel of the waves.

Consider a thin solid rod, cross-sectional area A, with Young's modulus Y, so that under a force F the rod extends by a fraction F/AY (note dimensions: elastic moduli all have units $ML^{-1}T^{-2}$, whereas F is MLT^{-2}). Let the density of the rod be ρ.

Suppose that at a point x along the rod an element of the rod (a thin disk) has been displaced by ξ as a result of wave passing down the rod (see figure T10.1). If at a point a little further along, at $x + \mathrm{d}x$, the displacement is $\xi + \mathrm{d}\xi$, then the element which was of length $\mathrm{d}x$ has been stretched. The amount of the stretch is

$$\mathrm{d}\xi = \frac{\partial \xi}{\partial x}\mathrm{d}x.$$

Remember that ξ is the *change* in position of a marker on the rod. The force at each point, then, is given by the local strain (change in length divided by length), i.e.

$$F(x) = AY\frac{\partial \xi}{\partial x}$$

If we had a rod under constant tension, of course, the *fractional* extension would be constant along the rod, and $\xi = \frac{F}{AY}x$.

If the amount of stretching is not constant along the rod, the force will not be constant either. In fact the force at $x + \mathrm{d}x$ will be

$$F' = F + \mathrm{d}F = F + \frac{\partial F}{\partial x}\mathrm{d}x$$

and the nett force on the element $\mathrm{d}x$ is therefore

$$F' - F = \frac{\partial F}{\partial x}\mathrm{d}x,$$

so that

$$F' - F = AY\frac{\partial^2 \xi}{\partial x^2}\mathrm{d}x.$$

The mass of the element of thickness $\mathrm{d}x$ and area A is $\rho A\mathrm{d}x$, and its change in position is given by ξ, so

$$\rho A\mathrm{d}x\frac{\partial^2 \xi}{\partial t^2} = AY\frac{\partial^2 \xi}{\partial x^2}\mathrm{d}x$$

or

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{Y}{\rho}\frac{\partial^2 \xi}{\partial x^2}. \tag{T10.1}$$

Thus waves of compression and expansion can travel along an elastic rod at a velocity $\sqrt{Y/\rho}$.

For steel, with $Y = 2\times10^{11}$ Pa, $\rho = 8000$ kg m^{-3}, we find $c = 5000$ m s^{-1}, a typical wave speed in a solid.

T10.2 Elastic waves in a bulk solid

In a bulk solid, there are two possible ways in which sound waves can travel: as compression waves (longitudinal) rather like the waves in the rod, or as shear waves (transverse). The expressions which relate the forces to the strains are rather more complex than for a gas, as a solid has a structure and compressing it along one axis is not the same as compressing in along all axes (if you compress a solid along one axis it will spread out along the other two: the rod is free to expand in this way, but the bulk solid is constrained so that it cannot) and this extra stiffness increases the speed of waves. The compression and shear wave speeds c_p and c_s are determined by two elastic moduli, bulk B and shear G.

Examples of waves in materials are shown in figure T10.2.

compression waves and shear waves

In each case the derivation is very similar to that for the waves in a rod:

- Consider an element of volume and its displacement (longitudinal or transverse)

- Calculate the difference in forces between the ends (compression or shear forces)
- Relate the force to the distortion ($F = (B + \frac{4}{3}G)A\frac{\partial\xi}{\partial x}$ or $F = GA\frac{\partial\xi}{\partial x}$)
- hence wave speeds
  - $c_\mathrm{p}^2 = (B + \frac{4}{3}G)/\rho$
  - $c_\mathrm{s}^2 = G/\rho$
  - typically, $c_\mathrm{p} \approx 2c_\mathrm{s}$
  - Earthquakes: different arrival times of different waves (but note that most of the damage comes from surface wave, which keep the energy of the quake close to the surface where it can damage buildings).

You should be able to derive the shear wave speed by following the 'recipe' above.

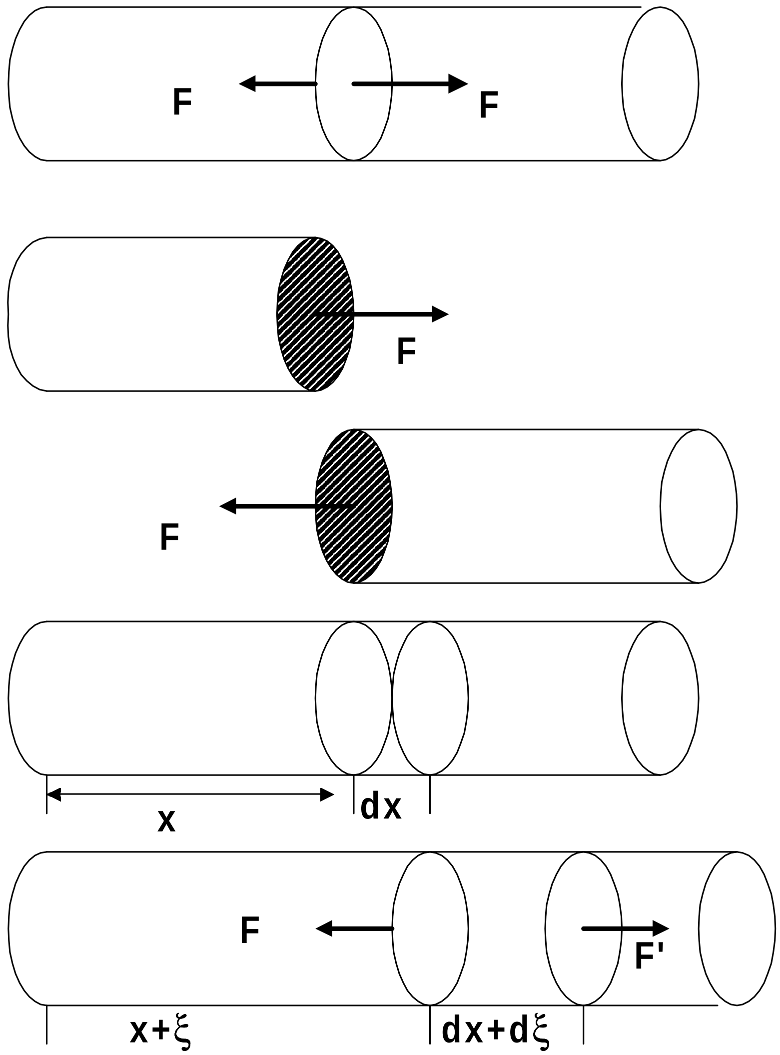

Figure T10.1: The displacement and associated force in a rod supporting a wave.

Figure T10.2: Waves in action: the earth's structure as revealed by the propagation of seismic waves; the destructive effects of an earthquake; ultrasound used for nondestructive testing of a pipe; the image of a foetus revealed by ultrasound.