Topic 9 - Waves in Finite Systems

normal modes, harmonics *FGT381, AF919-921*

So far we have talked about waves travelling along infinite strings. Current physics department budgets only run to strings of finite length. If we fix the ends of a string of length L, one end at $x = 0$ and the other at $x = L$, what sort of motions are possible? In mathematical terms, what is the effect of imposing *boundary conditions* on the motion?

If the frequency is fixed at ω then a general form for the displacement y at time t and position x would consist of two waves of unknown amplitudes a and b, which in general may be complex,

$$y(x,t) = ae^{i(\omega t + kx)} + be^{i(\omega t - kx)}.$$

We have chosen these forms so that the time variation of the two terms is the same, and we will be able to factor it out later. We know, though, that $y(x,t) = 0$ at $x = 0$ for all t, so

$$0 = (a + b)e^{i\omega t}$$

or

$$a = -b.$$

(If we think about the physical significance of this, it means that we have *equal* and *opposite* waves travelling in the two directions along the string.) Thus

$$\begin{aligned} y(x,t) &= ae^{i(\omega t + kx)} - ae^{i(\omega t - kx)} \\ &= ae^{i\omega t}\left[e^{ikx} - e^{-ikx}\right] \\ &= 2iae^{i\omega t}\sin(kx). \end{aligned}$$

We must now impose the boundary condition at the other end, $y = 0$ at $x = L$ for all t. Thus we must have

$$\sin(kL) = 0$$

or

$$kL = n\pi$$

where n is an integer. Writing this in terms of wavelength λ,

$$\frac{2\pi L}{\lambda} = n\pi$$

from which we deduce that only a certain set of wavelengths is allowed, those for which (labelling them with n)

$$\boxed{\lambda_n = \frac{2L}{n}}$$

that is, λ_n is $2L$, L, $2L/3$..., which have nodes (positions of zero displacement) at the ends of the string and 0, 1, 2... further nodes in between, with 1, 2, 3... antinodes (positions of maximum displacement) along its length.

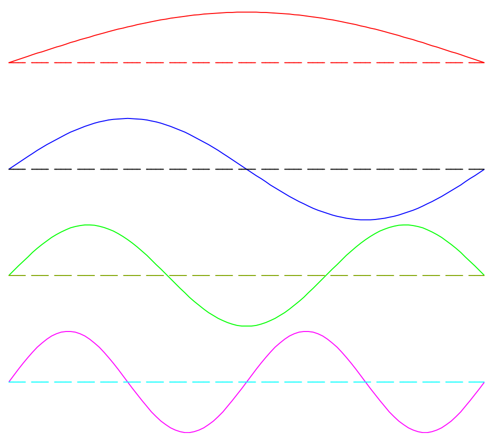

Figure T9.1: The displacements in the four lowest-frequency normal modes of transverse vibration of a stretched string with fixed ends.

These wavelengths are the *normal modes* on the string, and are sketched in figure T9.1. Instead of specifying the wavelengths, we could specify the frequencies,

$$\boxed{\omega_n = \frac{2\pi c}{\lambda_n} = \frac{n\pi c}{L}.}$$

These are the *normal frequencies*, *mode frequencies*, or *eigenfrequencies*. Note that they arise because of the *boundary conditions*. There are no limits on the frequencies which can be sent along a continuous string, but if we put limits on the motion we can only have certain frequencies.

nodes/antinodes of standing wave

Of course, all we have done to set up those standing waves is to superpose running waves of equal amplitude but opposite sign, travelling in opposite directions, but we have produced a disturbance in the general case $a = Ae^{i\phi}$ where A and ϕ determine the amplitude and phase

$$\begin{aligned} y(x,t) &= 2iae^{i\omega t}\sin(kx) \\ &\rightarrow -2A\sin(\omega t+\phi)\sin(kx) \end{aligned}$$

when we take the real part. That is, we have a wave where the zeros of displacement are always in the same place, and the amplitude simply varies sinusoidally with time.

Two equal and opposite running waves give a standing wave

Of course, we can *force* a string to vibrate at other frequencies: these natural frequencies are the only ones at which it will vibrate freely — that is, they are the ones which will satisfy the equation of motion with no applied force.

example - steel wire

As an example, take a steel wire which weighs 12.5 grammes per metre length. If this is put under a tension of 800 N (reasonable for a wire in a piano, say, and equivalent to the weight of a person), the sound velocity will be

$$\sqrt{\frac{T}{\rho}} = \sqrt{\frac{800}{12.5\times 10^{-3}}} = 253 \text{ m s}^{-1}.$$

Now suppose that the length of this wire is half a metre. The ends will be fixed (the case we treated above). The allowed wavelengths will therefore be 1 m, $\frac{1}{2}$ m, $\frac{1}{3}$ m, etc., and the corresponding frequencies are

$$\begin{aligned} f = \frac{\text{velocity}}{\text{wavelength}} &= 253 \text{ Hz} \\ & \quad 506 \text{ Hz} \\ & \quad 759 \text{ Hz}, \end{aligned}$$

which lie in the range of musical notes (concert A, for example, is 440 Hz).

wire with free ends

Fixed boundaries are not the only possibility. Instead suppose that the wire is stretched between supports so that the ends can slide vertically in frictionless grooves, so the the displacement is not zero but the transverse force is zero at the ends. We will just sketch the derivation of the normal modes for this situation.

As before, we start with a general solution

$$y(x,t) = ae^{i(\omega t+kx)} + be^{i(\omega t-kx)}.$$

Now, though, the transverse force[1] $T\partial y(x,t)/\partial x = 0$ and at $x = L$ for all t. Evaluating the derivative,

$$\frac{\partial yx,t}{\partial x} = ikae^{i(\omega t+kx)} - ikbe^{i(\omega t-kx)},$$

which gives at $x = 0$

$$0 = ik(a-b)Te^{i\omega t}$$

or

$$a = b.$$

Apart from the sign, this is the result we had before. Thus

$$\begin{aligned} y(x,t) &= ae^{i(\omega t+kx)} + ae^{i(\omega t-kx)} \\ &= ae^{i\omega t}\left[e^{ikx} + e^{-ikx}\right] \\ &= 2ae^{i\omega t}\cos(kx). \end{aligned}$$

We must now impose the boundary condition at the other end, $\partial y/\partial x = 0$ at $x = L$ for all t. Thus we must have, as before,

$$\sin(kL) = 0$$

leading to the same results

$$\lambda_n = \frac{2L}{n}.$$

That is, the allowed wavelengths are the same as before, but the patterns of displacement are different, as shown in figure T9.2.

[1]It is important to remember that it is the force that is zero. In this case this is the same as saying that the gradient of the string is zero, but it is important to think of the condition as applying to the force: otherwise, mistakes are likely to occur later when we deal with reflection and refraction, when we must match forces rather than gradients.

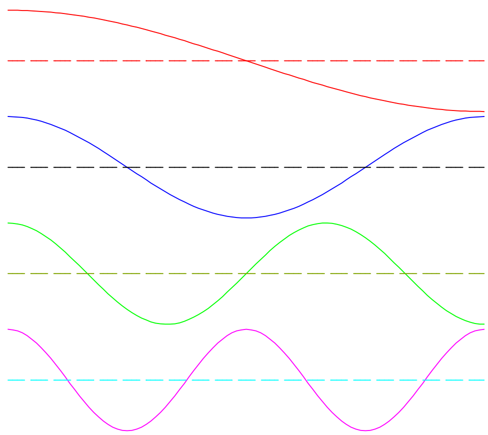

Figure T9.2: The displacements in the four lowest-frequency normal modes of transverse vibration of a stretched string with ends supported with no transverse force.

T9.1 Longitudinal and transverse waves

T9.2 Polarisation

Exactly as with the oscillator, we can have transverse oscillations in either of two directions, or we can have (as on a wave going down a spring, or sound in air) longitudinal motion. The direction of motion is perpendicular to the direction of travel of the wave we have *transverse polarisation*; if it is parallel we have *longitudinal polarisation*. If the motion of transverse polarisation stays in one plane, we have *plane polarisation*, but if there is a phase difference between the motions in the two directions we have, as for Lissajous figures, circular or elliptical polarisation.