# Topic 6 - Dispersion (continued)

### envelope function and carrier; phase and group velocities

A dispersive wave is characterised by a *carrier* travelling at the *phase velocity* $v_{\mathrm{p}}$ and an *envelope* travelling at the *group velocity* $v_{\mathrm{g}}$. The group velocity is the velocity at which a point of constant phase of the *amplitude modulation* travels, that is, it is the rate at which the *energy* in the wave is transmitted.

Phase velocity

$$v_{\mathrm{p}} = \frac{\omega}{k}$$

Group velocity

$$v_{\mathrm{g}} = \frac{\mathrm{d}\omega}{\mathrm{d}k}$$

Note that

$$v_g = \frac{\partial \omega}{\partial k} = \frac{\partial (v_p k)}{\partial k} = v_p + k\frac{\partial v_p}{\partial k}.$$

and usually (remember the refractive index) $v_p$ decreases with increasing $k$ (increases with increasing $\lambda$) so $v_g$ is less than $v_p$ – this is known as *normal dispersion*. There are circumstances in which $v_g > v_p$ (anomalous dispersion). This definition of what is normal and what is anomalous is taken from the behaviour of light. For light the phase velocity is related to the speed of light in the vacuum, $c$, through the refractive index, $n$, by $v_{\mathrm{p}} = c/n$, so

$$v_{\mathrm{g}} = \frac{c}{n} - k\frac{c}{n^2}\frac{\mathrm{d}n}{\mathrm{d}k}$$

and thus if $n$ increases with $k$ (increases with frequency, decreases with wavelength) we have normal dispersion. Typical experimental data are shown in figure T6.1.

### waves on the surface of a fluid

An example of a dispersive system is waves on the surface of a deep fluid. There are two mechanisms involved in such a wave, gravity and surface tension. The phase velocity is given by

$$v_p^2 = \frac{g}{k} + \frac{k\gamma}{\rho}$$

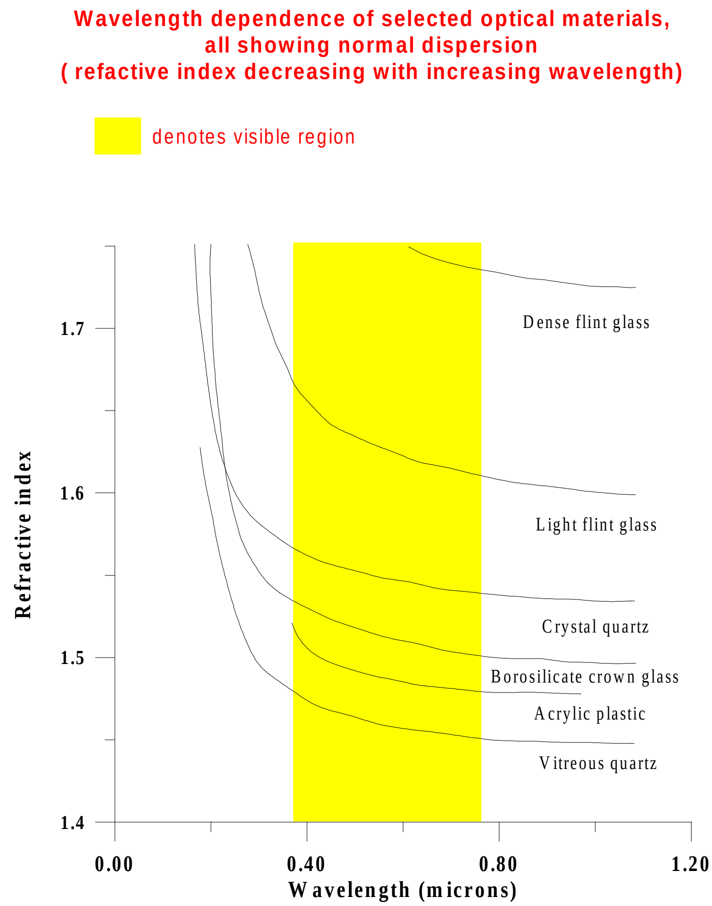


Figure T6.1: Typical variation of refractive index with frequency.

in terms of the acceleration due to gravity ($g$), surface tension ($\gamma$) and density($\rho$). Note that this means there is a *minimum* velocity as a function of $k$, when

$$2v_p\frac{\mathrm{d}v_p}{\mathrm{d}k} = \frac{-g}{k^2} + \frac{\gamma}{\rho}$$

which is zero when

$$k^2 = \frac{g\rho}{\gamma}.$$

For water, with $g = 9.8$, $\rho = 1000$ and $\gamma = 7.5 \times 10^{-2}$ N/m,

$$k = 361 \text{ m}^{-1}$$

or

$$\lambda = 2\pi/k = .017 \text{ m}.$$

Waves with both longer and shorter wavelengths than 17mm will propagate, being called *waves* and *ripples* respectively. For gravity waves (long wavelengths) the phase velocity increases with $\lambda$, see figure T6.2..

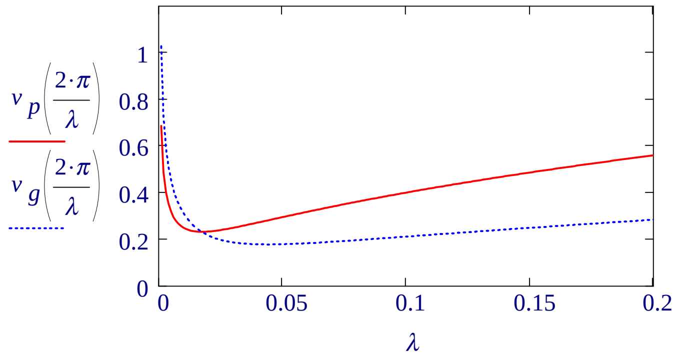


Figure T6.2: Dispersion of a surface wave on a deep fluid.

### Limiting cases

In the limit of very long wavelengths, $k \to 0$ and so

$$\begin{aligned}
v_{\rm p}^2 &\to \frac{g}{k} \\
v_{\rm p} &\to \sqrt{g}k^{-1/2} \\
v_{\rm g} &\to \sqrt{g}k^{-1/2} + k\frac{-1}{2}\sqrt{g}k^{-3/2} \\
&= \frac{1}{2}\sqrt{\frac{g}{k}} \\
&= \frac{1}{2}v_{\rm p}.
\end{aligned}$$

For short wavelengths (ripples), however

$$\begin{aligned}
v_{\rm p}^2 &\to \frac{\gamma}{\rho}k \\
v_{\rm p} &\to \sqrt{\frac{\gamma}{\rho}}k^{1/2}
\end{aligned}$$

$$
\begin{aligned}
v_{\mathrm{g}} &\to \sqrt{\frac{\gamma}{\rho}}k^{1/2} + k\frac{1}{2}\sqrt{\frac{\gamma}{\rho}}k^{-1/2} \\
&= \frac{3}{2}\sqrt{\frac{\gamma k}{\rho}} \\
&= \frac{3}{2}v_{\mathrm{p}}.
\end{aligned}
$$