Topic 5 - Single-Frequency Waves and Dispersion

For a frequency ω we want to form a function which contains the combination $x \pm ct$ and in which the time variation has the form $e^{i\omega t}$. We start by converting the combination $x - ct$, which has the dimensions of length, to a quantity with the dimensions of radians. We can do this if we multiply it by $2\pi/\lambda$, and take

$$\psi(x,t) = \psi_0 e^{i\frac{2\pi}{\lambda}(x-ct)}.$$

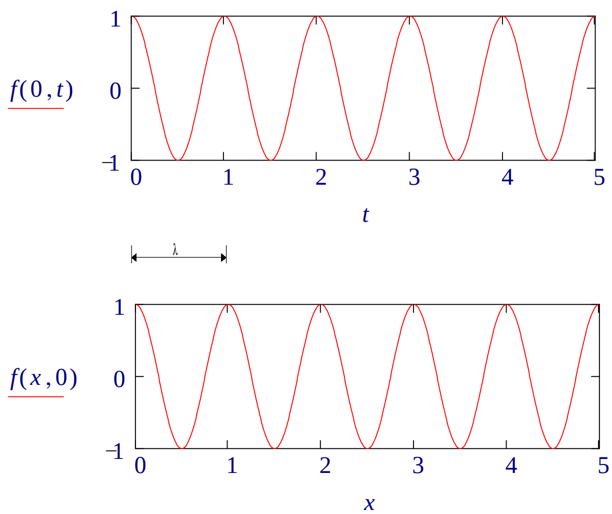

Figure T5.1: The propagation of a sinusoidal wave, the real part of $e^{i(kx-\omega t)}$, as a function of time (top) and position (bottom).

Why did we pick that particular combination of π and λ? The spacing between successive peaks in space (see figure T5.1) is λ, the wavelength, and in one wavelength the wave makes one complete cycle, so that when the position changes by one wavelength the phase changes by 2π. Similarly the spacing between successive peaks in time is λ/c, the period, so the frequency is c/λ, often called ν or f, the angular frequency ω is $2\pi c/\lambda$. The combination $2\pi/\lambda$ is called the wave vector, or wave number, often denoted by k.

We can write the general form of the wave in a number of equivalent ways:

$$
\begin{aligned}
\psi(x,t) &= ae^{i\frac{2\pi}{\lambda}(x-ct)} \\
&= ae^{i2\pi(\frac{x}{\lambda}-\nu t)} \\
&= ae^{i2\pi(\frac{x}{\lambda}-ft)} \\
&= ae^{i\omega(\frac{x}{c}-t)} \\
&= ae^{i(kx-\omega t)}
\end{aligned}
$$

As with the harmonic oscillator, we may choose to make a real, in which case we may need to include a phase shift in the exponent, or a may be complex, and itself include any phase shift, in which case we could write

$$a = Ae^{i\phi},$$

where A is the *real* amplitude and ϕ is the initial phase.

We have a series of relationships

$$c = \lambda\nu = \lambda f = \frac{\omega}{k}.$$

As we know the wave velocity may depend on the frequency, we have to be a little more precise about what we mean by the velocity.

The quantity we have defined by $c = \omega/k$ gives, in a sinusoidal wave, the speed at which peaks and troughs (points of constant phase) move through the medium - it is called the *phase velocity*.

linearity/superposition

We know that as the wave equation is linear, we may superpose solutions and still get a solution which is a solution of the wave equation.

running waves

The solutions which we shall from now on write as functions of the form $e^{i(kx+\omega t)}$ and $e^{-i(kx-\omega t)}$ are running waves - they travel along the x axis either left to right $(-)$ or right to left $(+)$. The direction of travel is determined by the *relative* sign of the x and t terms.

T5.1 Phase and group velocity *AF772-774*

Now that we are happy with the treatment of any arbitrary signal as a sum of sinusoidal signals, we are in a position to ask how a pulse might change in shape if the phase velocity is not constant, but depends on frequency.

Refractive index of materials

First, consider the physical reasons why a velocity might depend on frequency. Consider electromagnetic waves. The reason why the velocity of light in a material differs from that in free space is that the electromagnetic fields cause changes in the material. These may involve

- arranging molecules with permanent dipole moments (such as water)
- moving atoms about (an obvious example is an ionic crystal)
- moving electrons
  - valence electrons - near optical
  - core electrons - x-rays

The strongest changes in properties, as measured by refractive index

$$\text{refractive index} = \frac{\text{speed of light in free space}}{\text{speed of light in material}}$$

variously referred to as n or μ, occur near resonances - frequencies which coincide with intrinsic frequencies of the system (interatomic vibrations; electronic transitions). A schematic diagram of the variations is shown in figure T5.2.

An extreme example is the strong absorption of radiation at 18GHz (wavelength of 17mm) by reorienting water dipoles, being the basis of microwave cookery (or even microwave demolition of concrete). In fact, for various practical reasons, domestic ovens use a frequency of 2.45GHz.

simple two-frequency treatment

If we want to send signals, a single frequency is no use – it is a wave that is always there, and so conveys no information. To send a signal we need to turn it on and off, or at least to modulate its amplitude.

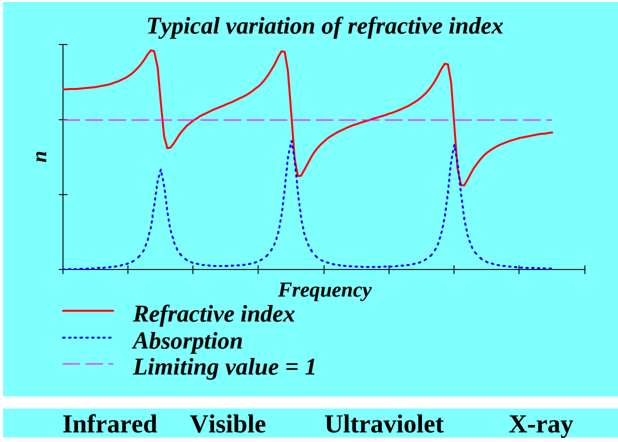

Figure T5.2: Schematic variation of refractive index and associated absorption with frequency.

A general treatment is complicated, but the essential details are captured by looking at a superposition of two waves of slightly different frequencies.

We know what such a superposition will look like, from our previous treatment of beats. Take a superposition of two waves, of equal amplitude:

$$\begin{aligned} \psi_1(x,t) &= a\cos(\omega_1 t - k_1 x) \\ \psi_2(x,t) &= a\cos(\omega_2 t - k_2 x) \end{aligned}$$

and suppose that the two contributing waves have *different* phase velocities (i.e. the velocity depends on the frequency)

$$\frac{\omega_1}{k_1} = v_{p1}$$

$$\frac{\omega_2}{k_2} = v_{p2}$$

. Now form the sum

$$\psi(x,t) = \psi_1(x,t)+\psi_2(x,t) = 2a\cos\left[\frac{\omega_1+\omega_2}{2}t - \frac{k_1+k_2}{2}x\right]\cos\left[\frac{\omega_1-\omega_2}{2}t - \frac{k_1-k_2}{2}x\right].$$

This is the product of two functions: the *carrier wave*

$$\cos\left[\frac{\omega_1+\omega_2}{2}t - \frac{k_1+k_2}{2}x\right]$$

has the average frequency of the two superposed waves, and travels at a speed, the *phase velocity*

$$v_p = \frac{\omega_1 + \omega_2}{k_1 + k_2} \approx \frac{\omega}{k}.$$

This wave is modulated by the envelope function

$$2a \cos \left[\frac{\omega_1 - \omega_2}{2} t - \frac{k_1 - k_2}{2} x \right].$$

What will be the velocity of the envelope function? It will move at

$$\frac{\omega_1 - \omega_2}{k_1 - k_2} = \frac{\Delta\omega}{\Delta k} \approx \frac{\mathrm{d}\omega}{\mathrm{d}k}$$

provided that ω_1 and ω_2, k_1 and k_2 are not too different.

envelope function and carrier

The component with the half-sum frequency is called the *carrier*, the half-difference wave is called the *envelope.*

The group velocity, as we have defined it here, is the velocity at which a point of constant phase of the *amplitude modulation* travels, that is, it is the rate at which a *signal* travels, or the rate at which the *energy* in the wave is transmitted.

As an example (figure T5.3) consider two waves, with $\omega_1 = 10\pi$, $k_1 = 10\pi$ and $\omega_2 = 11.5\pi$, $k_2 = 12\pi$, with phase velocities of 1 and $11.5/12 = 0.958$. The resultant has phase veolcity 0.977 and group velocity 0.75

Looking closely at the figure, one can see that whereas initially (at $t = 0$) the peak of the carrier wave coincided exactly with the peak of the envelope, at a later time the peak in the carrier has edged slightly ahead. By the time $t = 0.4$ (figure T5.4) the peak of the envelope corresponds with a minimum of the carrier.

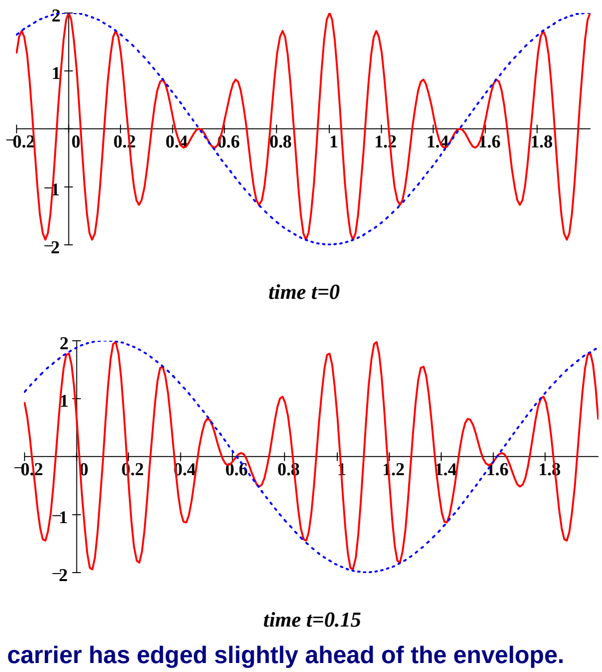

Figure T5.3: Two superposed sine waves, showing the carrier and envelope.

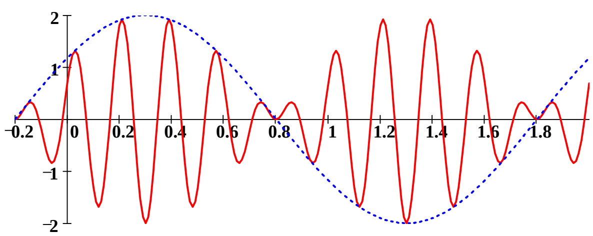

Figure T5.4: The same two superposed sine waves, showing the carrier and envelope at a later time.