Topic 3 – Combinations of Oscillations

In the last lecture we looked at the use of the complex exponential representation, and the way to handle it in computing energies. Now we want to look at the effects of superposing simple harmonic oscillations. An example might be a stretched string with a bead on it, which we drive with some form of magnetic coupling. The driving forces might be in or out of phase, of the same or different amplitudes, and might even have different frequencies. We assume, though, that the oscillations are continuous, and in a fixed phase relationship to one another.

L3.1 Superposition of two motions *AF200-208*

same frequency – same amplitude

The superposition of two oscillations may be treated in the phasor representation. We are adding together two vectors, both rotating with the same angular velocity. Thus the resultant will also be rotating with that angular velocity, as shown in figure T3.1.

Consider first two vectors of equal amplitude, a, with phase difference ϕ, as in figure T3.2.We use the cosine formula to calculate the amplitude:

$$\begin{aligned} A^2 &= a^2 + a^2 - 2a^2\cos(\pi - \phi) \\ &= 2a^2(1+\cos\phi) \\ &= 4a^2\cos^2(\phi/2) \\ A &= 2a\cos(\phi/2). \end{aligned}$$

Also, from the right-angled triangle in the diagram,

$$\begin{aligned} \sin\psi &= \frac{a\sin\phi}{A} \\ &= \frac{2a\sin(\phi/2)\cos(\phi/2)}{2a\cos(\phi/2)} \\ &= \sin(\phi/2) \\ \psi &= \phi/2. \end{aligned}$$

We could equally well (or, perhaps, more easily) use the complex exponential notation. The resultant motion is

$$\mathrm{Re}\left[Ae^{i(\omega t+\psi)}\right] = \mathrm{Re}\left[ae^{i\omega t} + ae^{i(\omega t+\phi)}\right]$$

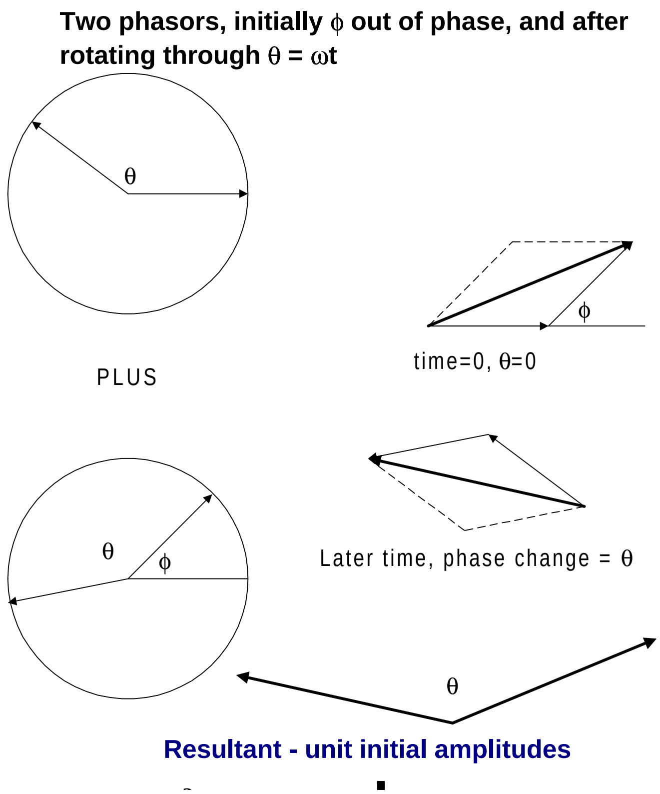

Figure T3.1: Addition of two phasors.

$$\begin{aligned} &= \mathrm{Re}\left[ae^{i(\omega t+\phi/2)}\left(e^{i\phi/2}+e^{-i\phi/2}\right)\right] \\ &= 2a\cos(\phi/2)\mathrm{Re}\left[e^{i(\omega t+\phi/2)}\right]. \end{aligned}$$

From this we immediately identify the amplitude as $2a\cos(\phi/2)$, and the phase ψ as $\phi/2$.

Many similar signals

We can use the phasor picture to consider what will happen when we add together several identical oscillatory signals, as in figure T3.3. The first point to note is that when there is no phase difference between them, if we add N identical signals we will obtain a total *amplitude* which is N times each individual amplitude. The *energy*, though, which is proportional to

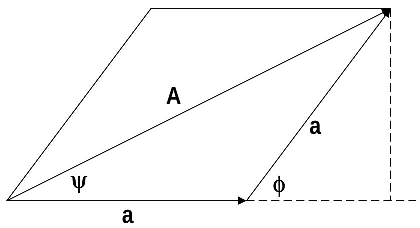

Figure T3.2: Addition of two phasors of equal amplitude.

the square of the amplitude, will now be N^2 times the original energy[1]. As the phase difference increases, though, the phasor diagram showing the addition starts to curl round on itself, eventually forming a closed loop, and the resultant gets shorter. For more vectors, as in figure T3.4, the resultant can be quite a small fraction of the maximum possible factor of N times the initial amplitude.

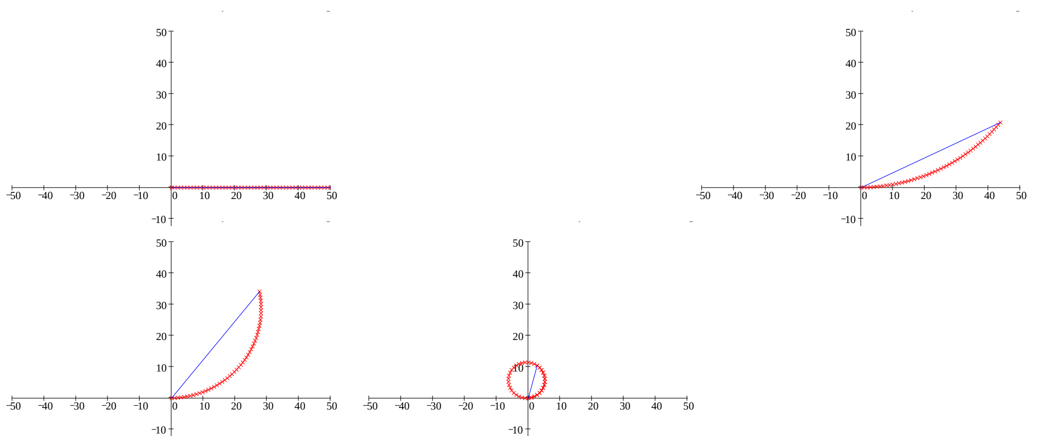

Figure T3.3: Addition of many phasors of equal amplitude: 50 phasors with relative phase difference 0 (top), 1 degree, 2 degrees and 10 degrees (bottom).

[1]It is instructive to think about the conservation of energy here. There is no problem in this situation where we are imagining some sort of force driving the oscillator — think about the work that is being done by each successive forcing function.

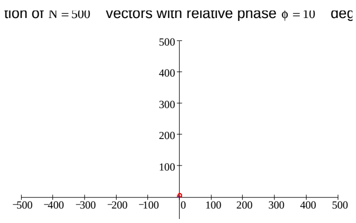

Figure T3.4: Addition of many phasors of equal amplitude: 500 phasors with relative phase difference 10 degrees.

Different frequencies – beats

We now consider the possibility that the signals being combined may have different frequencies.

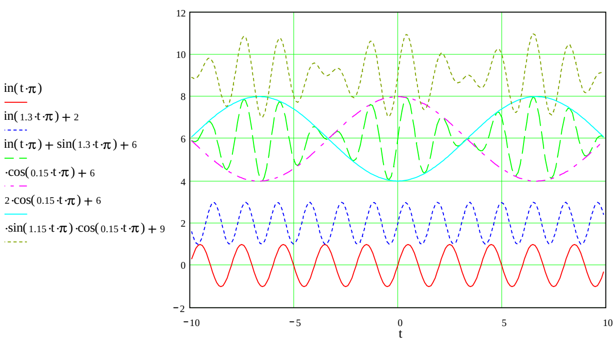

Figure T3.5: Superposition of two signals with different frequencies, showing the phenomenon of beats.

Suppose the signals have the same amplitude and phase, for example two unit amplitude sine signals.[2]

$$
\begin{aligned}
x_1(t) &= \mathrm{Re}\left[e^{i(\omega_1 t - \pi/2)}\right] \\
x_2(t) &= \mathrm{Re}\left[e^{i(\omega_2 t - \pi/2)}\right]
\end{aligned}
$$

[2]Note the trick used in this derivation, when we have the sum of two complex exponentials, of taking out a factor which is the average of the two, in order to obtain a cosine function. We will meet this way of simplifying expressions of this sort several times later in the course.

$$
\begin{aligned}
x_1(t) + x_2(t) &= \mathrm{Re}\left[e^{-i\pi/2}\left(e^{i\omega_1 t} + e^{i\omega_2 t}\right)\right] \\
&= \mathrm{Re}\left[e^{-i\pi/2} e^{i(\omega_1+\omega_2)t/2}\left(e^{i(\omega_1-\omega_2)t/2} + e^{-i(\omega_1-\omega_2)t/2}\right)\right] \\
&= 2\cos\frac{(\omega_1-\omega_2)t}{2}\mathrm{Re}\left[e^{-i\pi/2} e^{i(\omega_1+\omega_2)t/2}\right] \\
&= 2\cos\frac{(\omega_1-\omega_2)t}{2}\sin\frac{(\omega_1+\omega_2)t}{2}.
\end{aligned}
$$

This is the product of two functions, with half-sum and half-difference frequencies, and the resulting disturbance is shown in figure T3.5. The higher-frequency term is called the *carrier* frequency, the lower-frequency pattern which modulates the amplitude of the carrier is the *envelope*. This is the phenomenon of *beats*.

How will we perceive this effect? If we generate two sounds with slightly different frequencies, the ear will hear the high average frequency with its amplitude varying according to the envelope. This amplitude variation is called *beating*. What the ear will detect, though, the change in *amplitude* of the sound, which means that the beat frequency is the simply the *difference* in the two underlying frequencies. This is because a peak in the *amplitude*, that is the peak in the *loudness* of the sound, occurs twice in every period associated with $(\omega_1 - \omega_2)/2$. The diagram in figure T3.6 shows the modulus of the signal — note that it oscillates about a mean value, and the frequency of this oscillation, the beat frequency, is double the frequency of the envelope of the signal.

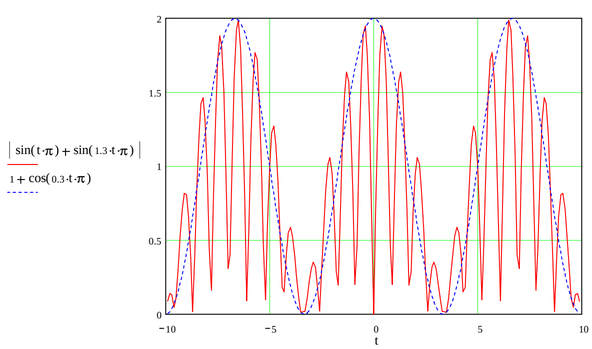

Figure T3.6: Superposition of two signals with different frequencies, giving beats. Here we are plotting the absolute value of the signal. The variation of the loudness has twice the frequency of the envelope function of the signal.

L3 Supplementary material: addition of phasors

Same frequency – different amplitude

The phasor diagram is shown in figure T3.7. Again we use the cosine rule to find

$$R^2 = a^2 + b^2 + 2ab\cos(\psi - \phi),$$

and take the ratio of the projections of the resultant onto the y and x axes to find the phase angle

$$\tan\theta = \frac{a\sin\phi + b\sin\psi}{a\cos\phi + b\cos\psi}.$$

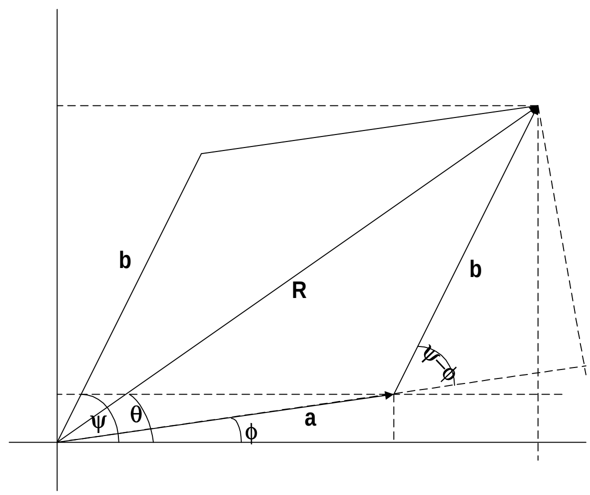

Figure T3.7: Addition of two phasors of different amplitudes.

Equally, we can use the complex notation, and recall

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(z_1^* + z_2^*) \\ &= |z_1|^2 + |z_2|^2 + (z_1 z_2^* + z_2 z_1^*) \\ &= |z_1|^2 + |z_2|^2 + 2\mathrm{Re}(z_1 z_2^*). \end{aligned}$$

So, with

$$\begin{aligned} z_1 &= ae^{i(\omega t + \phi)} \\ z_2 &= be^{i(\omega t + \psi)} \end{aligned}$$

where a and b are real, we find

$$|z_1 + z_2|^2 = a^2 + b^2 + 2ab\text{Re}\left[e^{i(\omega t+\phi)}e^{-i(\omega t+\psi)}\right].$$

Similarly, we know

$$\begin{aligned}\arg(z_1 + z_2) &= \tan^{-1}\left[\frac{\text{Im}(z_1 + z_2)}{\text{Re}(z_1 + z_2)}\right] \\ &= \tan^{-1}\left[\frac{a\sin\phi + b\sin\psi}{a\cos\phi + b\cos\psi}\right].\end{aligned}$$

Different directions (polarization)

Thinking back to the model we are using here, of a bead on a wire, there is no reason why the two oscillations should be in the same direction - we could drive the bead with magnetic fields at right angles, and thus drive it round a two-dimensional path. If the bead moves in a straight line, it is a *linearly polarized oscillator.*

Linear, circular, elliptical polarization

If, however, we drive the bead with the same amplitude and frequency along the x and y axes, but $\pi/2$ out of phase, we drive it round a circle - after all, this is just reconstructing the situation from which we derived the notion of the phasor. If the two motions are in phase, we drive it in a straight line at 45 degrees to the axis. Intermediate phase shifts result in various elliptical polarisations.

Lissajous figures

If the frequencies driving the x and y axes are different, we get the range of rather elegant figures known as Lissajous patterns.

Mathematics used in this topic

The Cosine rule: if a and b are two sides of a triangle with an angle θ between them, then the third side c is given by

$$c^2 = a^2 + b^2 - 2ab\cos(\theta).$$

For complex numbers

$$
\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)(z_1^* + z_2^*) \\
&= |z_1|^2 + |z_2|^2 + (z_1 z_2^* + z_2 z_1^*) \\
&= |z_1|^2 + |z_2|^2 + 2\mathrm{Re}(z_1 z_2^*).
\end{aligned}
$$

and

$$
\begin{aligned}
\arg(z_1 + z_2) &= \tan^{-1}\left[\frac{\mathrm{Im}(z_1 + z_2)}{\mathrm{Re}(z_1 + z_2)}\right] \\
&= \tan^{-1}\left[\frac{a\sin\phi + b\sin\psi}{a\cos\phi + b\cos\psi}\right].
\end{aligned}
$$