Radiation of Accelerated Charges

Non-relativistic Case

Consider a charge q which has always been at rest at the origin until time $t = 0$. The electric field lines clearly point away from the origin in all directions. At $t = 0$ we briefly apply a uniform acceleration which brings the velocity of the charge up to Δv in time Δt. At time $t > 0$, the charge will have moved a distance $t\Delta v$. Outside a circle of radius ct, the field lines cannot yet know that the charge has moved (no signal can move faster than light), so they point radially away from the origin. Presuming that $\Delta v \ll c$, the field lines point away from the *charge* within the annulus, which has width $c\Delta t$ (see figure).

Inside the annulus, the field lines must join up. This means there is a kink, which is propagating radially outward at speed c. This kink is nothing more than a pulse of radiation! We know that the electric field $\mathbf{E}$ of an electromagnetic wave is perpendicular to $\hat{\mathbf{k}}$, which means that it is only the θ-component of $\mathbf{E}$ which contributes to the flux. The Poynting flux for the EM pulse (for which $E = B$) is

$$\mathbf{S} = \frac{c}{4\pi} E_\theta^2 \hat{\mathbf{k}}.$$

To find E_θ is just a matter of simple geometry:

$$\frac{E_\theta}{E_r} = \frac{\Delta v \, t \sin\theta}{c\Delta t} = \frac{r\sin\theta}{c^2}\frac{\Delta v}{\Delta t},$$

where $r = ct$. Since E_r is just the Coulomb field,

$$E_\theta = \frac{q\sin\theta}{c^2 r}\frac{\Delta v}{\Delta t} = \frac{qa\sin\theta}{c^2} r.$$

Let W be the energy which is radiated. Then

$$S \equiv \frac{dW}{dt\; dA} = \frac{q^2 a^2 \sin^2\theta}{4\pi c^3 r^2}.$$

is the power radiated per unit area in a given direction. Integrating this over all area elements $r^2\, d\Omega$ on the surface of the annulus, we get the total radiated power:

$$P \equiv \frac{dW}{dt\; dA} = \frac{2q^2 a^2}{3c^3}.$$

This is known as Larmor's formula. A more careful treatment using the correct retarded potentials gives the same answer after considerably more work.

A couple of things to note about the radiation are:

- It is dipolar ($\propto \sin\theta$). If you are looking down the direction of acceleration, you don't see a kink in the electric field; the intensity is zero. The intensity is greatest if you are staring at the particle from above in the figure.
- The polarisation (the direction of $\mathbf{E}$) points along the direction of $\mathbf{a}$ projected onto the sphere of radius ct.
- We use the proper acceleration measured in the frame of the particle.

Spectrum

Integrating the Poynting flux over time, we find

$$\frac{dW}{dA} = \frac{c}{4\pi}\int_{-\infty}^{\infty} E^2(t)\, dt.$$

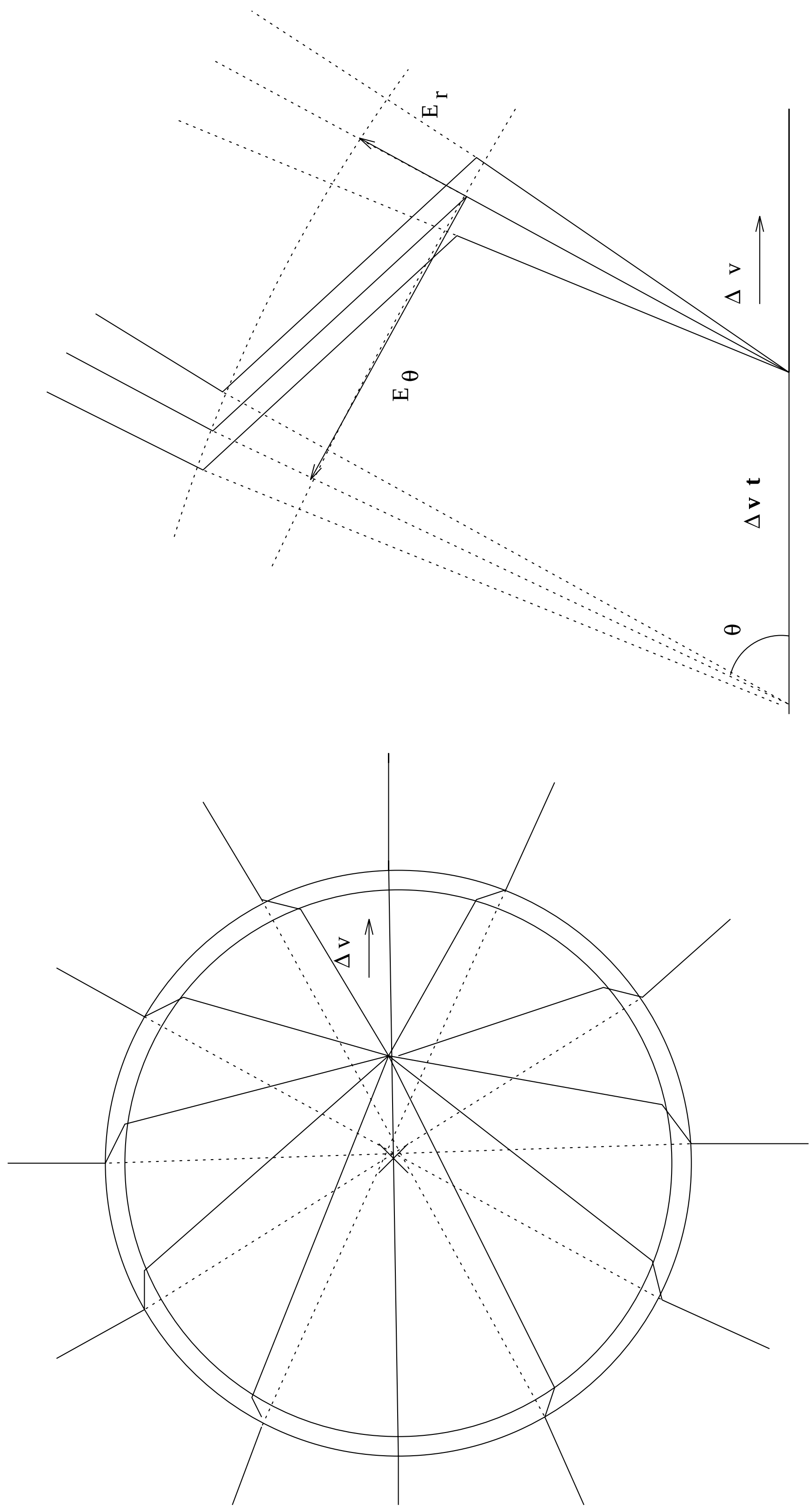

Figure 1: Figures showing the electric field lines from an accelerated charge. A pulse of radiation travels outward in the form of the kink in the field.

The spectral content of the radiated electric field is given by its Fourier transform:

$$\widetilde{E}_\nu = \int_{-\infty}^{\infty} E(t) e^{2\pi i \nu t}\, dt; \qquad E(t) = \int_{-\infty}^{\infty} \widetilde{E}_\nu e^{-2\pi i \nu t}\, d\nu.$$

Parseval's theorem allows us to write

$$\frac{dW}{dA} = \frac{c}{4\pi} \int_{-\infty}^{\infty} |\widetilde{E}_\nu|^2\, d\nu = \frac{c}{2\pi} \int_0^{\infty} |\widetilde{E}_\nu|^2\, d\nu.$$

The spectrum is therefore given in terms of the Fourier transform of the radiated electric field by

$$\frac{dW}{dA\, d\nu} = \frac{c}{2\pi} |\widetilde{E}_\nu|^2.$$

Relativistic Case

Since energy and time transform the same way under Lorentz transformations, dW/dt is a Lorentz invariant. Let the primed frame be the instantaneous rest frame of the particle. Then

$$P = P' = \frac{2q^2}{3c^3} \mathbf{a}'^2 = \frac{2q^2}{3c^3} (a_\parallel'^2 + \mathbf{a}_\perp'^2),$$

where $\mathbf{a}$ is written in terms of components parallel and perpendicular to the velocity. You can show that $a'_\parallel = \gamma^3 a_\parallel$ and $\mathbf{a}'_\perp = \gamma^2 \mathbf{a}_\perp$, which gives

$$P = \frac{2q^2}{3c^3} \gamma^4 (a_\perp^2 + \gamma^2 a_\parallel^2).$$

The exact form of the angular distribution of radiation is quite complicated, but the important thing to remember is that the radiation will be strongly *beamed* in the forward direction into a cone of opening angle $\sim 1/\gamma$. The faster the particle goes, the more the radiation is beamed in the forward direction.

References

Longair, *High Energy Astrophysics*, v. 1, Ch. 3.
Rybicki & Lightmann, *Radiative Processes in Astrophysics*, Ch. 2–4.
Shu, *The Physics of Astrophysics*, Ch. 15.