B.Sc. EXAMINATION

MAS 347 Mathematical Aspects of Cosmology

xxx, xx Sample Paper xx:xx-xx:xx
xx:xx-xx:xx

SOLUTIONS

SECTION A

Each question carries 10 marks. Attempt ALL questions.

1. *Explain briefly: (i) what is meant by the perfect cosmological principle and why it is wrong; (ii) what is meant by cosmological principle and why observational data support it. What is the approximate scale above which homogeneity is reached? Explain why the homogeneity of the Universe implies the existence of a cosmic time.*

A1. Solution.

Perfect cosmological principle: The Universe is the same every where and always is the same. The observational evidence that the Universe is evolving during its expansion rejects this principle.

Cosmological principle: The Universe is the same everywhere.

The expansion of the Universe, the evolution of objects in the Universe and the isotropy of Cosmic Microwave Background support the cosmological principle.

This scale is about 100 Mpc.

The homogeneity of the universe implies that the density depends only on time. The relation between the density and time,

$$\rho \sim t^{-2}$$

is the definition of the cosmic time.

2. *Use the Friedmann equation for a spatially flat Universe to find the Hubble constant H as a function of scale factor R if the present Universe contains dust with dimensionless density Ω_{d0} and radiation with dimensionless density Ω_{r0}.*

A2. Solution.

Taking into account that

$$\rho = \rho_d + \rho_r,$$

$$k = \Lambda = 0, \quad H = \frac{\dot{R}}{R} \text{ and } \rho_{cr} = \frac{3H_0^2}{8\pi G},$$

and that

$$\rho_d = \Omega_d \frac{3H_0^2}{8\pi G}\left(\frac{R_0}{R}\right)^3, \quad \rho_r = \Omega_r \frac{3H_0^2}{8\pi G}\left(\frac{R_0}{R}\right)^4,$$

the Friedmann equation can be written as

$$H^2 = \frac{8\pi G}{3} \cdot \frac{3H_0^2}{8\pi G}\left[\Omega_d \left(\frac{R_0}{R}\right)^3 + \Omega_r \left(\frac{R_0}{R}\right)^4\right],$$

hence

$$H^2 = H_0^2 \left[\Omega_d \left(\frac{R_0}{R} \right)^3 + \Omega_r \left(\frac{R_0}{R} \right)^4 \right].$$

At present moment

$$R = R_0, \quad H = H_0,$$

hence

$$\Omega_d + \Omega_r = 1, \quad \Omega_r = 1 - \Omega_d.$$

Finally

$$H = H_0 \left(\frac{R_0}{R} \right)^{3/2} \left[\Omega_d + \Omega_r \left(\frac{R_0}{R} \right) \right]^{1/2}.$$

[2/10]

[seen similar]

3. *Assume that a small fraction of the matter density in a spatially flat Universe can be explained by some hypothetical dark objects of mass $M = 10^{-3} M_\odot$ and that the dimensionless density of these objects is $\Omega_{obj} = 10^{-7}$. Estimate the average distance between these objects at present time and at redshift $z = 9$.*

A3. Solution.

$$\rho_{obj} = \Omega_{obj} \rho_{cr} (1+z)^3,$$

the number density is

$$n_{obj} = \frac{\Omega_{obj} \rho_{cr} (1+z)^3}{M_{obj}},$$

the average distance between these objects is determined from

$$d_{obj}^3 n_{obj} \approx 1,$$

[seen similar]

hence

$$d_{obj} = n_{obj}^{-1/3} = \left(\frac{M_{obj}}{\Omega_{obj} \rho_{cr}} \right)^{1/3} (1+z)^{-1} =$$

$$= \left(\frac{10^{-3} M_\odot}{10^{-7} \times 10^{-27} \text{ kg m}^{-3}} \right)^{1/3} (1+z)^{-1} =$$

$$= \left(\frac{10^{-3} \times 2 \times 10^{30} \text{ kg}}{10^{-7} \times 10^{-27} \text{ kg m}^{-3}} \right)^{1/3} (1+z)^{-1} =$$

$$= \left(2 \times 10^{61} \right)^{1/3} \text{ m}(1+z)^{-1} \approx 3 \times 10^{20} \text{ m}(1+z)^{-1} \approx 10 \text{ kpc}(1+z)^{-1}.$$

At $z = 0$, $d_{obj} = 10$ kpc and at $z = 9$, $d_{obj} = 1$ kpc.

[Next question overleaf.]

4. *Give the definition of the covariant tensor of the fourth rank, A_{iklm}, and the contravariant tensor of the third rank, B^{ikl}. In the local Galilean frame $x^i_{[G]}$ of reference a mixed tensor of the second rank, C^i_k has the only one non-vanishing component, $C^0_{0[G]} = 1$, and all other components are equal to zero. Write down all components of this mixed tensor in arbitrary frame of reference, x^i, in terms of the transformation matrices $\alpha^l_{m[G]} = \frac{\partial x^l}{\partial x^m_{[G]}}$ and $\beta^l_{m[G]} = \frac{\partial x^l_{[G]}}{\partial x^m}$.*

A4. Solution.

The covariant tensor of the fourth rank is transformed as

$$A_{iklm} = \tilde{S}^n_i \tilde{S}^p_k \tilde{S}^v_{l\tilde{S}^j_m} A'_{npvj},$$

and this is the definition of the fourth rank covariant tensor.

The contravariant tensor of the third rank is transformed as

$$B^{ikl} = S^i_n S^k_m S^l_j B'^{nmj},$$

and this is the definition of the third rank contravariant tensor.

The mixed tensor of the second rang is transformed as

$$C^i_k = S^i_n \tilde{S}^j_k C'^n_j.$$

If C'^n_j is in the local galilean coordinates, then

$$C'^n_j = \delta^n_0 \delta^0_j$$

and

$$S^l_m = \alpha^l_{m[G]}$$

and

$$\tilde{S}^l_m = \beta^l_{m[G]},$$

we have

$$C^i_k = \alpha^i_{m[G]} \beta^l_{k[G]} C'^m_l = \alpha^i_{m[G]} \beta^l_{k[G]} \delta^m_0 \delta^0_l = \alpha^i_{0[G]} \beta^0_{k[G]}.$$

5. *Write the Hubble law in vector form. Consider three galaxies in an expanding Universe located at points A, B and C. Prove that if the Hubble law is valid for an observer at A, then it is also valid for observers at B and C. Assume that the vector $\vec{r}_{AB}$ is perpendicular to the vector $\vec{r}_{AC}$. For an observer in galaxy A, galaxy B has a redshift $z^B_{(A)} = 0.6$ and galaxy C has a redshift with $z^C_{(A)} = 0.8$. Find the redshift $z^C_{(B)}$ of galaxy C, measured by an observer in galaxy B.*

A5. Solution.

The Hubble law in vector form is

$$\vec{v} = H\vec{r}.$$

Using this vector form, we have for the observer in a

$$\vec{v}_{ab} = H\vec{r}_{ab}, \quad \vec{v}_{ac} = H\vec{r}_{ac},$$

Hence for the observer in b we have

$$\vec{v}_{ba} = -\vec{v}_{ab} = -H\vec{r}_{ab} = H\vec{r}_{ba},$$

and

$$\vec{v}_{bc} = \vec{v}_{ba} + \vec{v}_{ac} = -\vec{v}_{ab} + \vec{v}_{ac} = -H\vec{r}_{ab} + H\vec{r}_{ac} = H(-\vec{r}_{ab} + \vec{r}_{ac}) = H(\vec{r}_{ba} + \vec{r}_{ac}) = H\vec{r}_{bc}.$$

Then, for the observer c we have

$$\vec{v}_{ca} = -\vec{v}_{ac} = -H\vec{r}_{ac} = H\vec{r}_{ca},$$

and

$$\vec{v}_{cb} = -\vec{v}_{bc} = -H\vec{r}_{bc} = H\vec{r}_{cb}.$$

Redshif z is related with velocity as

$$z = \frac{v}{c} = \frac{Hr}{c},$$

Hence

$$\left[z^c_{(b)}\right]^2 = \left(\frac{H}{c}\right)^2 [r_{cb}]^2 = \left(\frac{H}{c}\right)^2 \left[[r_{ca}]^2 + [r_{ab}]^2\right] = \left(\frac{H}{c}\right)^2 [r_{ac}]^2 + \left(\frac{H}{c}\right)^2 [r_{ab}]^2 = \left[z^c_{(a)}\right]^2 + \left[z^b_{(a)}\right]^2 =$$

$$= (0.8)^2 + (0.6)^2 = 0.64 + 0.36 = 1,$$

hence $z^c_{(b)} = 1$.

 [Next section overleaf.]

SECTION B

Each question carries 25 marks. You may attempt all questions but, except for a bare pass, only marks for the best TWO questions will be counted.

1. *Assume that the Universe is open ($k = -1$) with $\Lambda = 0$ and contains only dust.*

*[**15 Marks**] (a) Using the Friedmann and energy conservation equations, verify that the evolution of the scale factor can be expressed in the following parametric form:*

$$R(\eta) = \frac{\beta}{2}(\cosh\eta - 1), \quad t(\eta) = \frac{\beta}{2c}(\sinh\eta - \eta),$$

where β is a constant.

*[**10 Marks**] (b) Determine β in terms of present Hubble constant H_0 and density parameter Ω_0.*

B1. Solution.

(a) The energy conservation equation in the case $\alpha = 0$, which means $p = \alpha\rho c^2 = 0$, gives $d(\rho R^3) = 0$,

hence $\rho = \rho_0(R_0/R)^3$.

Substituting this result to the Friedman equation we have

$$\dot{R}^2 = c^2\left(\gamma/R + 1\right),$$

where

$$\gamma = 8\pi\rho_0 R_0^3/3c^2.$$

Then we calculate $\dot{R}^2$, using the parametric solution:

$$\dot{R}^2 = \left[\frac{d\left[\frac{\beta}{2}(\cosh\eta - 1)\right]}{d\left[\frac{\beta}{2c}(\sinh\eta - \eta)\right]}\right]^2 = c^2\frac{\sinh^2\eta}{(\cosh\eta - 1)^2} = c^2\frac{1 + \cosh\eta}{\cosh\eta - 1}.$$

Putting this into the Friedman equation we have

$$c^2\frac{1 + \cosh\eta}{\cosh\eta - 1} = c^2\left(\frac{2\gamma}{\beta(\cosh\eta - 1)} + 1\right), \quad 1 + \cosh\eta = \frac{2\gamma}{\beta} + 1 + \cosh\eta,$$

so we see that this parametric solution does satisfies the Friedman equation, if

$$\beta = \gamma = 8\pi\rho_0 R_0^3/3c^2.$$

(b) From the Friedman equation, taken at the moment t_0, we have

$$H_0^2 R_0^2 = H_0^2\Omega_0 + c^2,$$

so we can express R_0 in terms of H_0 and Ω_0 as

$$R_0 = \frac{c}{H_0\sqrt{1-\Omega_0}}.$$

Then substituting this to the formula for β we have

$$\beta = \frac{c\Omega_0}{H_0(1-\Omega_0)^{3/2}}.$$

2. *A spatially flat Universe contains dust with dimensionless density $\Omega_{d0} = 0.3$ and dark energy corresponding to cosmological constant Λ.*

*[**15 Marks**] (a) Calculate the present deceleration parameter q_0 if the Universe started to accelerate when the density of dust was 64 times larger than at present.*

*[**10 Marks**] (b) Determine the redshift when the contribution of "dark energy" to the density of the Universe was only 1%.*

B2. Solution.

(a)

Since the Universe is flat $k = 0$.

Since it contains only dust and dark energy in Λ-term form, $p = 0$.

$$q = -\frac{R\ddot{R}}{\dot{R}^2} = \frac{4\pi G\rho R^2}{3\dot{R}^2} - \frac{\Lambda R^2}{3\dot{R}^2} = \frac{1}{H^2}\left[\frac{4\pi G\rho_{d0}}{3}\left(\frac{R_0}{R}\right)^3 - \frac{\Lambda}{3}\right].$$

At the present moment

$$q_0 = \frac{1}{H^2}\left[\frac{4\pi G\rho_{d0}}{3} - \frac{\Lambda}{3}\right].$$

$$\rho_{d0} = \Omega_d\rho_{cr} = \Omega_d\frac{3H_0^2}{8\pi G}.$$

[seen similar]

At the moment when $q = 0$

$$\frac{4\pi G\rho_{d0}}{3}\left(\frac{R_0}{R}\right)^3 = \frac{\Lambda}{3}.$$

Given that

$$\left(\frac{R_0}{R}\right)^3 = 64,$$

hence

$$\frac{\Lambda}{3} = \frac{4\pi G}{3}\rho_{d0} \times 64,$$

thus

$$q_0 = \frac{1}{H^2}\frac{4\pi G\rho_{d0}}{3}(1-64) = -\frac{63}{3H_0^2} \times \frac{4\pi G}{3}\Omega_d\frac{3H_0^2}{8\pi G} = -10.5 \times 0.3 = -3.15.$$

(b)

$$H^2 = \frac{8\pi G}{3}(\rho_d + \frac{\Lambda}{8\pi G},$$

hence the contribution of Λ-term into total density, as we could see in question A4, is

$$\rho_\Lambda = \frac{\Lambda}{8\pi G}.$$

Let

$$\alpha = \frac{\rho_\Lambda}{\rho_d + \rho_\Lambda},$$

then

$$\rho_d = \frac{\rho_\Lambda(1-\alpha)}{\alpha}.$$

taking into account that

$$\frac{R_0}{R} = 1 + z,$$

we have

$$\rho_{d0}(1+z)^3 = \frac{\rho_\Lambda(1-\alpha)}{\alpha}.$$

Given that

$$\rho_{d0} = 0.3\rho_{cr} \;\; \rho_\Lambda = 0.7\rho_{cr},$$

hence taking $\alpha = .01$, we have

$$z = -1 + \left[\frac{0.7 \times 0.99}{0.3 \times 0.01}\right]^{1/3} \approx 5.$$

3. *Consider a sphere in a Robertson-Walker model with comoving coordinate $\chi = \chi_s$.*

*[**6 Marks**] (a) Using the Robertson-Walker metric, find (i) the physical radius $r(\chi_s, t)$ of the sphere at time t and (ii) the circumference $C(\chi_s, t)$ of a circle in the plane $\theta = \pi/2$. Express your results in terms of the scale factor $R(t)$ and χ_s.*

*[**19 Marks**] (b) A spherical galaxy of diameter D has redshift z and apparent angular diameter $\Delta\theta$. Show that*

$$\Delta\theta = \frac{D\sqrt{k}(1+z)}{R_0 \sin(\sqrt{k}\chi)},$$

where R_0 is present scale factor. Assuming that equation of state parameter $\alpha = 0$, $\Lambda = 0$ and $k = 0$, use the equation for radially propagating photons to determine an integral relationship between z and χ.

B3. Solution.

(a) From the Robertson-Walker metric

$$r = \int_0^{\chi_s} ds|_{dt=0,\, d\theta=0,\, d\phi=0} = R\int_0^{\chi_s} d\chi = R\chi_s.$$

From the Robertson-Walker metric

$$C = \int_0^{2\pi} ds|_{dt=0,\, d\theta=0,\, d\chi=0} = R\frac{\sin(\sqrt{k}\chi)}{\sqrt{k}}\int_0^{2\pi} d\phi = 2\pi R\frac{\sin(\sqrt{k}\chi)}{\sqrt{k}}.$$

(b)

$$\Delta\theta = 2\pi\frac{D}{C} = \frac{2\pi D}{2\pi R\frac{\sin(\sqrt{k}\chi}{\sqrt{k}}} = \frac{D\sqrt{k}}{R\sin(\sqrt{k}\chi)} = \frac{D\sqrt{k}(1+z)}{R_0\sin(\sqrt{k}\chi)}.$$

We see that the we should just relate χ with z. For radially propagating photons $ds = 0$, $d\theta = 0$ and $d\phi = 0$.

So from the Robertson-Walker metric we have $(cdt)^2 - (Rd\chi)^2 = 0$ or $d\chi = \pm cdt/R$.

Choosing sign "-" corresponding to photons propagating inward we have

$$\chi = -c\int_{t_0}^{t_z} dt/R,$$

where t_z is the moment of emission determined by z. Then

$$\chi = -c\int_{R_0}^{R_z}\frac{dR}{HR^2} = c\int_{R_z}^{R_0}\frac{dR}{HR^2}.$$

From the Friedman equation we have

$$H^2R^2 = H_0^2\Omega_0 R_0^3 R^{-1} - kc^2$$

and for the present time

$$H_0^2R_0^2 = H_0^2\Omega_0 R_0^2 - kc^2,$$

hence

$$HR = H_0R_0\sqrt{\Omega_0\frac{R_0}{R} + (1-\Omega_0)}.$$

Then taking into account that

$$R(z) = R_0/(1+z)$$

and

$$dR/R = -dz/(1+z),$$

we have

$$\chi = \frac{c}{H_0 R_0}\int_0^z \frac{dx}{(1+x)\sqrt{1+\Omega_0 x}}.$$

In the case $k = 0$ and $\Omega_0 = 1$ we have

$$\chi = \frac{c}{H_0 R_0}\int_0^z \frac{dx}{(1+x)^{3/2}} = \frac{2c}{H_0 R_0}\left(1-(1+z)^{-1/2}\right),$$

and

$$\Delta\theta = \frac{DH_0}{4\pi c}\frac{(1+z)^{3/2}}{(1+z)^{1/2}-1}.$$

4. *Consider a dust sphere of average density ρ' in a background flat Universe with $k = \Lambda = 0$. Consider the amplitude of the small density perturbation*

$$\delta(R) = \frac{\rho'(R)-\rho(R)}{\rho(R)}$$

as a function of scale factor R, where $\rho(R)$ is the average density of the Universe.

*[**15 Marks**] (a) Show that the equation for evolution of $\delta(R)$ can be written in the form*

$$\frac{d^2\delta}{dR^2} + \frac{3}{2R}\frac{d\delta}{dR} - \frac{3}{2R^2}\delta = 0.$$

[Hint: Show first that $(R'-R)/R = -\delta/3$.]

*[**10 Marks**] (b) Show that the general solution of this equation can be represented in terms of two independent modes, one of which is growing, while the other is decaying. Given that these two modes were equal at redshift $z = 9$, find their ratio at the present moment.*

B4. Solution

(a) Starting from

$$\ddot{R} = -\frac{4\pi G\rho R}{3},$$

perturb R and ρ: $R' = R(1+h)$, and $\rho' = \rho(1+\delta)$.

 [This question continues overleaf . . .]

To relate h and δ we use the conservation of energy equation $\rho R^3 = \rho R^3(1+3h)(1+\delta)$, or $1 = 1 + 3h + \delta$, so $h = -\delta/3$.

[1/15]

Hence

$$R' = R(1 - \frac{\delta}{3}),$$

$$\dot{R}' = \dot{R}\frac{dR'}{dR},$$

$$\ddot{R}' = \ddot{R}\frac{dR'}{dR} + \dot{R}^2\frac{d^2R'}{dR^2},$$

Putting this in the perturbed equation

$$\ddot{R}' = -\frac{4\pi G\rho' R'}{3},$$

we have

$$\ddot{R}\frac{d}{dR}[R(1-\frac{\delta}{3})] + \dot{R}^2\frac{d^2}{dR^2}[R(1-\frac{\delta}{3})] = -\frac{4\pi G\rho R}{3}(1+\delta)(1-\frac{\delta}{3}),$$

taking into account unperturbed equation

$$\ddot{R} = -\frac{4\pi G\rho R}{3}$$

[1/15]and unperturbed Ftiedmann equation

$$\dot{R}^2 = \frac{8\pi G\rho R^2}{3},$$

in first order with respect to δ we obtain

$$\frac{4\pi G\rho R}{3}\left\{-\frac{d}{dR}[R(1-\frac{\delta}{3})] + 2R\frac{d^2}{dR^2}[R(1-\frac{\delta}{3})] + 1 + \delta - \frac{\delta}{3}\right\} = 0,$$

thus

$$\delta - R\frac{d\delta}{dR} - \frac{2}{3}R^2\frac{d^2\delta}{dR^2} = 0,$$

finally

$$\frac{d^2\delta}{dR^2} + \frac{3}{2R}\frac{d\delta}{dR} - \frac{3}{2R^2}\delta = 0.$$

(b) Taking trial solution

$$\delta = AR^m,$$

we obtain

$$m(m-1) + \frac{3m}{2} - \frac{3}{2} = 0, \quad 2m^2 + m - 3 = 0.$$

Solutions of this quadratic equation are

$$m_{\pm} = \frac{1}{2}\left(-\frac{1}{2} \pm \sqrt{\frac{1}{6} + 4}\right) = \frac{-1 \pm 5}{4},$$

thus $m_+ = 1$ and $m_- = -\frac{3}{2}$ (growing and decaying modes). So we have

$$\delta = A_+(R/R_0) + A_-(R/R_0)^{-3/2}.$$

Taking into account that

$$R/R_0 = (1+z)^{-1},$$

at the moment when $z = 9$ and $\frac{R}{R_0} = 0.1$

$$0.1A_+ \approx 30A_-,$$

hence at the present moment the ratio of growing mode to decaying is

$$\frac{A_+}{A_-} \approx 300.$$

[End of examination paper.]