Queen Mary University of London

B.Sc. EXAMINATION

MAS 347 Mathematical Aspects of Cosmology

xxx, xx Sample Paper xx:xx-xx:xx
Duration: 2 hours

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions at the beginning of each Section. The use of an electronic calculator is permitted but no graph-plotting facilities may be used. Please state on your answer book the name and type of machine used.

Physical Constants

Gravitational constant	G	6.7×10^{-11} N m^2 kg^{-2}
Speed of light	c	3×10^8 m s^{-1}
Solar mass	$M_\odot$	2.0×10^{30}kg
Gravitational radius of Sun	$r_{g\odot}$	3 km
Hubble constant	H_0	70 km s^{-1}Mpc^{-1}
Hubble radius	c/H_0	6×10^3 Mpc

1 pc $= 3.1 \times 10^{16}$ m

NOTATION

Three-dimensional tensor indices are denoted by Greek letters $\alpha, \beta, \gamma, ...$ and take on the values 1, 2, 3.

Four-dimensional tensor indices are denoted by Latin letters $i, k, l, ...$ and take on the values 0, 1, 2, 3.

The metric signature $(+ - - -)$ is used.

Partial derivatives are denoted by ",".

Covariant derivatives are denoted by ";".

USEFUL FORMULAS.

Cosmology

$$ds^2 = c^2dt^2 - R^2(t)\left[d\chi^2 + \frac{\sin^2(\sqrt{k}\chi)}{k}(d\theta^2 + \sin^2\theta d\phi^2)\right] \quad \text{(Robertson–Walker metric)},$$

$$\ddot{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)R + \frac{\Lambda R}{3} \quad \text{(acceleration equation)}$$

$$q = -\frac{\ddot{R}R}{\dot{R}^2} \quad \text{deceleration parameter}$$

$$\dot{R}^2 + kc^2 = \frac{8\pi G}{3}\rho R^2 + \frac{\Lambda R^2}{3} \quad \text{(Friedmann equation)}$$

$$d(\rho c^2 V) = -pdV \quad \text{(energy conservation equation)}$$

$$\rho_{crit} = \frac{3H_0^2}{8\pi G} = 0.92 \times 10^{-26} \text{ kg m}^{-3} \quad \text{(critical density)}$$

General Relativity

Minkowski metric:

$$ds^2 = \eta_{ik}dx^i dx^k = c^2dt^2 - dx^2 - dy^2 - dz^2$$

Covariant derivatives:

$$A^i_{;k} = A^i_{,k} + \Gamma^i_{km}A^m, \quad A_{i;k} = A_{i,k} - \Gamma^m_{ik}A_m, \text{ where } \Gamma^i_{kn} \text{ are Christoffel symbols}$$

Christoffel symbols:

$$\Gamma^i_{kl} = \frac{1}{2}g^{im}\left(g_{mk,l} + g_{ml,k} - g_{kl,m}\right)$$

Geodesic equation:

$$\frac{du^i}{ds} + \Gamma^i_{kn}u^k u^n = 0,$$

where

$u^i = dx^i/ds$ is the 4-velocity along the geodesic.

Riemann tensor:

$$A^i_{;k;l} - A^i_{;l;k} = -A^m R^i_{mkl}, \quad \text{where } R^i_{klm} = g^{in}R_{nklm},$$

$$R^i_{klm} = \Gamma^i_{km,l} - \Gamma^i_{kl,m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl}.$$

Symmetry properties of the Riemann tensor:

$$R_{iklm} = -R_{kilm} = -R_{ikml}, \quad R_{iklm} = R_{lmik}.$$

Bianchi identity:

$$R^n_{ikl;m} + R^n_{imk;l} + R^n_{ilm;k} = 0.$$

Ricci tensor:

$$R_{ik} = g^{lm} R_{limk} = R^m_{imk}.$$

Scalar curvature:

$$R = g^{il} g^{km} R_{iklm} = g^{ik} R_{ik} = R^i_i.$$

Einstein equations:

$$R^i_k - \frac{1}{2}\delta^i_k R = \frac{8\pi G}{c^4} T^i_k,$$

where T^i_k is the Stress-Energy tensor.

Gravitational radius:

$$r_g = 2GM/c^2 = 3(M/M_\odot)\ \text{km}, \quad \text{where} \ \ M_\odot \ \ \text{is the mass of the Sun.}$$

SECTION A

Each question carries 10 marks. Attempt ALL questions.

1. Explain briefly: (i) what is meant by the perfect cosmological principle and why it is wrong; (ii) what is meant by cosmological principle and why observational data support it. What is the approximate scale above which homogeneity is reached? Explain why the homogeneity of the Universe implies the existence of a cosmic time.

2. Use the Friedmann equation for a spatially flat Universe to find the Hubble constant H as a function of scale factor R if the present Universe contains dust with dimensionless density Ω_{d0} and radiation with dimensionless density Ω_{r0}.

3. Assume that a small fraction of the matter density in a spatially flat Universe can be explained by some hypothetical dark objects of mass $M = 10^{-3} M_\odot$ and that the dimensionless density of these objects is $\Omega_{obj} = 10^{-7}$. Estimate the average distance between these objects at present time and at redshift $z = 9$.

4. Give the definition of the covariant tensor of the fourth rank, A_{iklm}, and the contravariant tensor of the third rank, B^{ikl}.

In the local Galilean frame $x^i_{[G]}$ of reference a mixed tensor of the second rank, C^i_k has the only one non-vanishing component, $C^0_{0[G]} = 1$, and all other components are equal to zero. Write down all components of this mixed tensor in arbitrary frame of reference, x^i, in terms of the transformation matrices $\alpha^l_{m[G]} = \frac{\partial x^l}{\partial x^m_{[G]}}$ and $\beta^l_{m[G]} = \frac{\partial x^l_{[G]}}{\partial x^m}$.

5. Write the Hubble law in vector form. Consider three galaxies in an expanding Universe located at points A, B and C. Prove that if the Hubble law is valid for an observer at A, then it is also valid for observers at B and C. Assume that the vector $\vec{r}_{AB}$ is perpendicular to the vector $\vec{r}_{AC}$. For an observer in galaxy A, galaxy B has a redshift $z^B_{(A)} = 0.6$ and galaxy C has a redshift with $z^C_{(A)} = 0.8$. Find the redshift $z^C_{(B)}$ of galaxy C, measured by an observer in galaxy B.

SECTION B

Each question carries 25 marks. You may attempt all questions but, except for a bare pass, only marks for the best TWO questions will be counted.

1. Assume that the Universe is open ($k = -1$) with $\Lambda = 0$ and contains only dust.

[**15 Marks**] (a) Using the Friedmann and energy conservation equations, verify that the evolution of the scale factor can be expressed in the following parametric form:

$$R(\eta) = \frac{\beta}{2}(\cosh\eta - 1), \quad t(\eta) = \frac{\beta}{2c}(\sinh\eta - \eta),$$

where β is a constant.

[**10 Marks**] (b) Determine β in terms of present Hubble constant H_0 and density parameter Ω_0.

2. A spatially flat Universe contains dust with dimensionless density $\Omega_{d0} = 0.3$ and dark energy corresponding to cosmological constant Λ.

[**15 Marks**] (a) Calculate the present deceleration parameter q_0 if the Universe started to accelerate when the density of dust was 64 times larger than at present.

[**10 Marks**] (b) Determine the redshift when the contribution of "dark energy" to the density of the Universe was only 1%.

3. Consider a sphere in a Robertson-Walker model with comoving coordinate $\chi = \chi_s$.

[**6 Marks**] (a) Using the Robertson-Walker metric, find (i) the physical radius $r(\chi_s, t)$ of the sphere at time t and (ii) the circumference $C(\chi_s, t)$ of a circle in the plane $\theta = \pi/2$. Express your results in terms of the scale factor $R(t)$ and χ_s.

[**19 Marks**] (b) A spherical galaxy of diameter D has redshift z and apparent angular diameter $\Delta\theta$. Show that

$$\Delta\theta = \frac{D\sqrt{k}(1+z)}{R_0 \sin(\sqrt{k}\chi)},$$

where R_0 is present scale factor. Assuming that equation of state parameter $\alpha = 0$, $\Lambda = 0$ and $k = 0$, use the equation for radially propagating photons to determine an integral relationship between z and χ.

[This question continues overleaf . . .]

4. Consider a dust sphere of average density ρ' in a background flat Universe with $k = \Lambda = 0$. Consider the amplitude of the small density perturbation

$$\delta(R) = \frac{\rho'(R) - \rho(R)}{\rho(R)}$$

as a function of scale factor R, where $\rho(R)$ is the average density of the Universe.

[**15 Marks**] (a) Show that the equation for evolution of $\delta(R)$ can be written in the form

$$\frac{d^2\delta}{dR^2} + \frac{3}{2R}\frac{d\delta}{dR} - \frac{3}{2R^2}\delta = 0.$$

[Hint: Show first that $(R' - R)/R = -\delta/3$.]

[**10 Marks**] (b) Show that the general solution of this equation can be represented in terms of two independent modes, one of which is growing, while the other is decaying. Given that these two modes were equal at redshift $z = 9$, find their ratio at the present moment.

[End of examination paper.]