Lecture 10. *Last updated 14.04.10*

X. IN VICINITY OF THE SCHWARZSCHILD BLACK HOLE

A. Test particles in the Schwarzschild Metric

Taking into account the spherical symmetry of the Schwarzschild metric we can choose our spherical coordinates in such a way that the plane of orbit coincides with the equatorial plane $\theta = \pi/2$. Then the Hamilton−Jacobi equation in the Schwarzschild metric can be written as

$$\left(1 - \frac{r_g}{r}\right)^{-1} \left(\frac{\partial S}{c\partial t}\right)^2 - \left(1 - \frac{r_g}{r}\right)\left(\frac{\partial S}{\partial r}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial \phi}\right)^2 - m^2c^2 = 0. \tag{X.1}$$

Since all coefficients in this equation do not depend on t and ϕ we can say that

$$\frac{\partial S}{\partial t} = -E, \text{ and } \frac{\partial S}{\partial \phi} = L, \tag{X.2}$$

where E and L are constants, which by definition are the energy and angular momentum of the particle under consideration. Then putting

$$S = -Et + L\phi + S_r(r) \tag{X.3}$$

into the Hamilton−Jacobi equation we have

$$\left(1 - \frac{r_g}{r}\right)^{-1} \frac{E^2}{c^2} - \left(1 - \frac{r_g}{r}\right)\left(\frac{dS_r(r)}{dr}\right)^2 - \frac{L^2}{r^2} - m^2c^2 = 0, \tag{X.4}$$

hence,

$$\frac{dS_r(r)}{dr} = \left(1 - \frac{r_g}{r}\right)^{-1/2} \sqrt{\left(1 - \frac{r_g}{r}\right)^{-1}\frac{E^2}{c^2} - \frac{L^2}{r^2} - m^2c^2} =$$

$$= \left(1 - \frac{r_g}{r}\right)^{-1} \sqrt{\frac{E^2}{c^2} - \left(1 - \frac{r_g}{r}\right)\left(\frac{L^2}{r^2} + m^2c^2\right)}. \tag{X.5}$$

Then the contravariant components of the four-momentum are

$$p^0 \equiv mc\frac{dx^0}{ds} = mc\frac{cdt}{ds} = g^{00}p_0 = \left(1 - \frac{r_g}{r}\right)^{-1} \frac{\partial S}{c\partial t} = -\frac{E}{c}\left(1 - \frac{r_g}{r}\right)^{-1}, \tag{X.6}$$

$$p^1 \equiv mc\frac{dx^1}{ds} = mc\frac{dr}{ds} = g^{11}p_1 = -\left(1-\frac{r_g}{r}\right)\frac{\partial S}{\partial r} =$$

$$= -\left(1-\frac{r_g}{r}\right)^{1/2}\sqrt{\left(1-\frac{r_g}{r}\right)^{-1}\frac{E^2}{c^2}-\frac{L^2}{r^2}-m^2c^2} = -\sqrt{\frac{E^2}{c^2}-\left(1-\frac{r_g}{r}\right)\left(\frac{L^2}{r^2}+m^2c^2\right)}, \tag{X.7}$$

$$p^3 \equiv mc\frac{dx^3}{ds} = mc\frac{d\phi}{ds} = g^{33}p_3 = -\frac{1}{r^2}\frac{\partial S}{\partial \phi} = -\frac{L}{r^2}. \tag{X.8}$$

Then we can rewrite above equations as

$$\frac{dt}{ds} = -\frac{E}{mc^3}\left(1-\frac{r_g}{r}\right)^{-1}, \tag{X.9}$$

$$\frac{dr}{ds} = -\frac{1}{mc^2}\sqrt{E^2-U_{eff}^2}, \tag{X.10}$$

$$\frac{d\phi}{ds} = -\frac{L}{mcr^2}, \tag{X.11}$$

where

$$U_{eff} = mc^2\sqrt{\left(1+\frac{L^2}{m^2c^2r^2}\right)\left(1-\frac{r_g}{r}\right)} \tag{X.12}$$

is called the "effective potential energy". For given radius U_{eff} is equal to the energy of a particle which has the turn point ($\frac{dr}{d\phi}=0$), i.e. Apastron or Periastron, for this r. Indeed

$$\frac{dr}{d\phi} = \frac{mc}{Lr^2}\sqrt{E^2-U_{eff}^2}, \tag{X.13}$$

hence, if

$$\frac{dr}{d\phi} = 0, \text{ then } U_{eff} = E. \tag{X.14}$$

Thus the condition

$$E > U_{eff} \tag{X.15}$$

determines the admissible range of the motion. The effective potential includes potential energy plus kinetic energy of non-radial motion, in the relativistic manner; this kinetic energy is determined by angular momentum L.

B. Stable and Unstable Circular Orbits

The radius of the stable circular orbit is obtained from the simultaneous solution of the equations

$$U_{eff} = E \tag{X.16}$$

and

$$\frac{dU_{eff}}{dr} = 0. \tag{X.17}$$

From Eq.(X.17) we have

$$dU_{eff}^2/du = 0, \tag{X.18}$$

where $u = 1/r$. Hence,

$$-r_g\left(1+\frac{L^2u^2}{m^2c^2}\right)+(1-r_gu)\frac{2L^2u}{m^2c^2}=0, \ \ or \ \ r_gr^2+3r_g\left(\frac{L}{mc}\right)^2-2\left(\frac{L}{mc}\right)^2r=0. \tag{X.19}$$

Solving this equation we have

$$r_{\pm}=\frac{L^2}{m^2c^2r_g}\pm\sqrt{\left(\frac{L^2}{m^2c^2r_g}\right)^2-\frac{3L^2}{m^2c^2}}=\frac{L^2}{m^2c^2r_g}\left(1\pm\sqrt{1-\frac{3r_g^2m^2c^2}{L^2}}\right). \tag{X.20}$$

The larger root corresponds to the stable orbit. One can see that

$$1-\frac{3r_g^2m^2c^2}{L^2}>0. \tag{X.21}$$

Hence,

$$-\sqrt{3}mcr_g \le L \le \sqrt{3}mcr_g. \tag{X.22}$$

Substituting

$$L=\sqrt{3}mcr_g \tag{X.23}$$

into equation for the radius of circular orbits (X.20), we have for the radius of the last stable orbit

$$r_{lso}=3r_g. \tag{X.24}$$

C. Propagation of light in the Schwarzschild metric

Let me remind you that for photons

$$ds=0. \tag{X.25}$$

We can introduce some scalar parameter λ varying along world line of the light signal and introduce then a vector

$$k^i=\frac{dx^i}{d\lambda}, \tag{X.26}$$

which is tangent to the word line. This vector is called four- dimensional wave vector. Then

$$ds^2=g_{ik}dx^idx^k=g_{ik}k^ik^kd\lambda^2=0 \tag{X.27}$$

and we have

$$k_ik^i=g^{ik}k_ik_k=0. \tag{X.28}$$

Substituting covariant vector

$$k_i=-\frac{\partial\psi}{\partial x^i}, \tag{X.29}$$

where ψ is a scalar, we obtain the Eikonal Equation in gravitational field

$$g^{ik}\Psi_{,i}\Psi_{,k}=0. \tag{X.30}$$

The physical meaning of Ψ (called the Eikonal follows from

$$\Psi = -\int k_i dx^i, \tag{X.31}$$

which looks like the phase of electromagnetic wave. If the Eikonal equation is solved, one can obtain the world line of photon:

$$\frac{dx^i}{d\lambda} \equiv k^i = g^{in} k_n = -g^{in} \Psi_{,n}. \tag{X.32}$$

In the equatorial plane of a Schwarzschild black hole the solution of the Eikonal equation can be written in the form

$$\Psi = -\omega t + \frac{b\omega}{c}\phi + \Phi_r(r), \tag{X.33}$$

whereω is the frequency of the photon and b is its impact parameter.
Substituting this expression to the Eikonal equation we obtain

$$\frac{1}{1-\frac{r_g}{r}}\frac{\omega^2}{c^2} - \frac{1}{r^2}\left(\frac{b\omega}{c}\right)^2 - \left(1-\frac{r_g}{r}\right)(-p_1)^2 = 0, \tag{X.34}$$

where

$$p_1 \equiv p_r = -\Psi_{,1} = -\frac{d\Phi_r(r)}{dr} = \pm\sqrt{\frac{1}{1-\frac{r_g}{r}}\left[\frac{1}{1-\frac{r_g}{r}}\frac{\omega^2}{c^2} - \frac{b^2\omega^2}{c^2 r^2}\right]}. \tag{X.35}$$

One can easily show that photons can move along unstable circular orbits given by

$$U_{eff(ph)} = 1, \text{ and } \frac{dU_{eff(ph)}}{dr} = 0, \tag{X.36}$$

where $U_{eff(ph)}$ plays the role of the effective potential for photons and is given by

$$U_{eff(ph)} = \frac{b^2}{r^2}\left(1-\frac{r_g}{r}\right). \tag{X.37}$$

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