Lecture 8. *Last updated 07.03.10*

VIII. SOLVING EFES

A. Weak field and slow motion approximation

In small velocity approximation

$$T_i^k \approx \rho c^2 u_i u^k, \tag{VIII.1}$$

where ρ is the mass density, i.e., $T_0^0 = \rho c^2$ and all other components are small, i.e., $|T_\alpha^0| \ll T_0^0$ and $|T_\alpha^\beta| \ll T_0^0$. This means that $T \equiv T_i^i \approx T_0^0$.
In weak field approximation one can neglect by the non-linear part in the Ricci tensor:

$$R_{00} = R_0^0 \approx \Gamma^\alpha_{00,\alpha} = -\frac{1}{2}\eta^{\alpha\beta} g_{00,\alpha,\beta} = \frac{1}{c^2}\phi_{,\alpha,\beta}, \tag{VIII.2}$$

where ϕ is defined by

$$g_{00} = 1 - \frac{2\phi}{c^2}. \tag{VIII.3}$$

Following usual notations

$$\eta^{\alpha\beta} g_{00,\alpha,\beta} = \triangle g_{00}, \tag{VIII.4}$$

where $\triangle$ is the Laplace operator. From EFEs we obtain

$$R_0^0 = \frac{1}{c^2}\triangle\phi = \frac{8\pi G}{c^4}(T_0^0 - \frac{1}{2}T) \approx \frac{8\pi G}{c^4}(T_0^0 - \frac{1}{2}T_0^0) = \frac{4\pi G}{c^4}T_0^0. \tag{VIII.5}$$

Hence,

$$\triangle\phi = 4\pi G\rho. \tag{VIII.6}$$

This is the Poisson equation, hence, as one can see, in this approximation EFEs give the Newtonian gravity and ϕ is the Newtonian gravitational potential.

B. The Schwarzschild metric as an exact solution of EFEs

Let r, θ, ϕ are spherical space coordinates. The most general spherically symmetric gravitational field can be described by the interval in the following form

$$ds^2 = h(r,t)dr^2 + k(r,t)(sin^2\theta d\phi^2 + d\theta^2) + l(r,t)dt^2 + a(r,t)drdt. \tag{VIII.7}$$

By transformations of coordinates

$$r = f_1(r', t'), t = f_2(r', t'), \tag{VIII.8}$$

we always can make

$$a(r,t) = 0 \text{ and } k(r,t) = -r^2. \tag{VIII.9}$$

Thus

$$ds^2 = e^{\nu}c^2dt^2 - r^2(sin^2\theta d\phi^2 + d\theta^2) - e^{\lambda}dr^2. \tag{VIII.10}$$

Taking into account that

$$g_{00} > 0 \text{ and } g_{11} < 0, \tag{VIII.11}$$

we can see that

$$g_{00} = e^{\nu}, \;\; g_{11} = -e^{\lambda}, \;\; g_{22} = -r^2, \text{ and } g_{33} = -r^2\sin^2\theta \tag{VIII.12}$$

$$g^{00} = e^{-\nu}, \;\; g^{11} = -e^{-\lambda}, \;\; g^{22} = -r^{-2} \text{ and } g^{33} = -r^{-2}\sin^{-2}\theta. \tag{VIII.13}$$

Now we can calculate the Christoffell symbols:

$$\Gamma^1_{11} = \frac{\lambda'}{2}, \;\; \Gamma^0_{10} = \frac{\nu'}{2}, \;\; \Gamma^2_{33} = -\sin\theta\cos\theta, \;\; \Gamma^0_{11} = \frac{\lambda}{2}e^{\lambda-\nu}, \tag{VIII.14}$$

$$\Gamma^1_{22} = -re^{-\lambda}, \;\; \Gamma^1_{00} = \frac{\nu}{2}e^{\nu-\lambda}, \;\; \Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r}, \;\; \Gamma^3_{23} = \cot\theta, \tag{VIII.15}$$

$$\Gamma^0_{00} = \frac{\dot{\nu}}{2}, \;\; \Gamma^1_{10} = \frac{\dot{\lambda}'}{2}, \;\; \Gamma^1_{33} = -r\sin^2\theta e^{-\lambda}, \tag{VIII.16}$$

where $'$ means partial derivative with respect to r. Then after straightforward calculations of the components of the Ricci tensor we obtain the Einstein's equations:

$$\frac{8\pi G}{c^4}T^1_1 = -e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2}, \tag{VIII.17}$$

$$\frac{8\pi G}{c^4}T^2_2 = \frac{8\pi G}{c^4}T^3_3 =$$

$$= -\frac{1}{2}e^{-\lambda}\left(\nu'' + \frac{\nu'^2}{2} + \frac{\nu' - \lambda'}{r} - \frac{\nu'\lambda'}{2}\right) + \frac{1}{2}e^{-\nu}\left(\ddot{\lambda} + \frac{\dot{\lambda}^2}{2} - \frac{\dot{\lambda}\dot{\nu}}{2}\right), \tag{VIII.18}$$

$$\frac{8\pi G}{c^4}T^0_0 = -e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2}, \tag{VIII.19}$$

$$\frac{8\pi G}{c^4}T^1_0 = -e^{-\lambda}\frac{\dot{\lambda}}{r}. \tag{VIII.20}$$

In vacuum, where all $T^i_k = 0$, we have

$$- e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} = 0, \tag{VIII.21}$$

$$e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 0, \tag{VIII.22}$$

$$\dot{\lambda} = 0, \tag{VIII.23}$$

The most unpleasant equation fortunately is not independent and follows from other three equations. One can proove this by straightforward calculations or by using the Bianchi identity. From equation (VIII.23) follows that $\lambda = \lambda(r)$, i.e. does not depend on t. From equations (VIII.21) and (VIII.22) follows that

$$\lambda' + \nu' = 0, \text{ hence } \lambda + \nu = f(t). \tag{VIII.24}$$

Now we can use our last freedom in coordinate transformation, namely we can transform the time coordinate, $t = f(t')$ to make $f(t) = 0$. As a result we obtain

$$e^{-\lambda} = e^{\nu}. \tag{VIII.25}$$

Thus we actually proved a very important theorem: If a gravitational field is spherical symmetric then this field is static! Now the system has been reduced to the single equation (VIII.22), which after multiplying by r^2 can be written as

$$e^{-\lambda}(r\lambda' - 1) + 1 = 0 \text{ or } -\left(e^{-\lambda}r\right)' + 1 = 0. \tag{VIII.26}$$

Finally

$$e^{-\lambda} = e^{\nu} = 1 + \frac{A}{r}, \tag{VIII.27}$$

where A is a constant of integration. One can see that if $r \to \infty$, then

$$e^{-\lambda} = e^{\nu} \to 1, \tag{VIII.28}$$

which corresponds to the Minkowskian space-time.
In order to determine the constant A let consider a test particle far from the centre of gravitating object. It's radial acceleration is given by the geodesic equation:

$$\frac{d^2r}{ds^2} + \Gamma^1_{ik}u^iu^k = 0. \tag{VIII.29}$$

If we assume that the particle moves slowly, i.e. four-velocity $u^i \approx \delta^i_0$ and $ds \approx cdt$ we obtain

$$\frac{d^2r}{dt^2} \approx -c^2\Gamma^1_{ik}\delta^i_0\delta^k_0 = -c^2\Gamma^1_{00} =$$

$$= -\frac{c^2}{2}g^{1n}(g_{0n,0} + g_{n0,0} - g_{00,n}) = -\frac{c^2}{2}g^{11}(g_{01,0} + g_{10,0} - g_{00,1}) \approx -\frac{c^2}{2}\frac{dg_{00}}{dr} =$$

$$= -\frac{c^2}{2}\frac{de^{-\lambda}}{dr} = -\frac{c^2}{2}\frac{d}{dr}(1 + \frac{A}{r}) = \frac{Ac^2}{2r^2}. \tag{VIII.30}$$

On other hand we know from Newtonian theory that

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2}, \tag{VIII.31}$$

hence the constant of integration

$$A = -\frac{2Gm}{c^2} = -r_g \text{ and } g_{00} = 1 - \frac{r_g}{r}, \tag{VIII.32}$$

where r_g is the so called gravitational radius

$$r_g = \frac{2Gm}{c^2}. \tag{VIII.33}$$

Finally we derived the famous solution of the EFEs obtained by K. Schwarzschild in 1916, the same year when Einstein published his equations. This solution is called the Schwarzschild metric:

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - r^2(sin^2\theta d\phi^2 + d\theta^2) - \frac{dr^2}{1 - \frac{r_g}{r}}. \tag{VIII.34}$$

One can see that this metric describes a curved space-time. To prove, for example, that even the space itself is curved, let us compare the physical radial distance, l, with the corresponding circumference, C. In the flat Euclidian space

$$l = \frac{C}{2\pi}, \tag{VIII.35}$$

while in the case of the Schwarzschild metric

$$dl^2 = \frac{dr^2}{1 - \frac{r_g}{r}} + r^2(\sin^2\theta d\phi^2 + d\theta^2), \tag{VIII.36}$$

hence

$$l = \int_{r_1}^{r_2} \frac{dr}{\sqrt{1 - \frac{r_g}{r}}} > r_2 - r_1 = \frac{l_{circl_2} - l_{circl_1}}{2\pi}. \tag{VIII.37}$$

One can see also that time runs at a different rate at different radii, indeed

$$d\tau = \sqrt{g_{00}}dt = \sqrt{1 - \frac{r_g}{r}}dt. \tag{VIII.38}$$

C. Physical singularity versus coordinate singularity in the Schwarzschild metric

We can prove that there is no physical singularity at $r = r_g$. For that we produce the following transformation of coordinates

$$c\tau = \pm ct \pm \int \frac{f(r)dr}{1 - \frac{r_g}{r}}, \tag{VIII.39}$$

$$R = ct + \int \frac{dr}{\left(1 - \frac{r_g}{r}\right) f(r)}, \tag{VIII.40}$$

where $f(r)$ is an arbitrary function. Now the interval can be written in the following form:

$$ds^2 = \frac{1 - \frac{r_g}{r}}{1 - f^2}(c^2d\tau^2 - f^2dR) - r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{VIII.41}$$

To eliminate "singularity" at $r = r_g$, we can choose $f(r)$ in such a way that $f(r_g) = 1$. For example,

$$f(r) = \sqrt{\frac{r_g}{r}}. \tag{VIII.42}$$

In this case

$$R - c\tau = \int \frac{(1 - f^2)dr}{\left(1 - \frac{r_g}{r}\right) f} = \int \sqrt{\frac{r}{r_g}} dr = \frac{2}{3} \frac{r^{3/2}}{r_g^{1/2}} \tag{VIII.43}$$

and

$$r = \frac{3}{2}(R - c\tau)^{2/3} r_g^{1/3}, \tag{VIII.44}$$

$$ds^2 = c^2 d\tau^2 - \frac{dR^2}{\left[\frac{3}{2r_g}(R - c\tau)\right]^{2/3}} - \left[\frac{3}{2}(R - c\tau)\right]^{4/3} r_g^{2/3}(d\theta^2 + sin^2\theta d\varphi^2). \tag{VIII.45}$$

We can see that there is now singularity at $r = r_g$, indeed if $r = r_g$

$$\frac{3}{2}(R - c\tau) = r_g. \tag{VIII.46}$$

In other words, the formal "singularity" ar $r = r_g$ can be removed by the transformation of coordinates.
The real physical singularity does take place at $r = 0$ when, say, the scalar curvature is infinite (one can easily verify this by straightforward calculations) and this fact can not be removed by any transformation of coordinates.

Back to Content **Previous Lecture** **Next Lecture**