Lecture 5. *Last updated 04.02.10*

V. MOTION OF A TEST PARTICLE IN A GRAVITATIONAL FIELD

A. Hamilton-Jacobi equation

Any object of a small enough mass is called a test particle. Small mass means that gravitational field generated by this object is negligible in comparison with the external gravitational field generated by other, much more massive, objects. The role of such test particle can be played by a planet around a star or a star around a massive black hole, or by photon propagating around a neutron star or black hole.
From the previous lecture we know that the motion of particles and photons in a given gravitational field is described by the space-time geodesics. The geodesic equations are very useful for physical understanding of the motion of particles and propagation of photons; however, it is easier to work with the Hamilton−Jacobi equation. The advantage of this approach is that it equates the motion of particles with the propagation of waves.
The derivation of Hamilton−Jacobi equation is really very simple. From the definition of the four-velocity

$$u^i = \frac{dx^i}{ds}, \tag{V.1}$$

we have

$$ds^2 = g_{ik}dx^i dx^k = g_{ik}u^i u^k ds^2 = u_i u^i ds^2, \tag{V.2}$$

hence

$$u^i u_i = 1. \tag{V.3}$$

Four-momentum of the particle is defined as

$$p^i = mcu^i, \text{ hence } p_i p^i = g^{ik} p_i p_k = m^2 c^2. \tag{V.4}$$

Taking into account that a covariant vector transforms as the gradient of a scalar, we can introduce such a scalar function that

$$p_i = -\frac{\partial S}{\partial x^i}, \tag{V.5}$$

then we immediately obtain the Hamilton−Jacobi Equation for a particle in a gravitational field

$$g^{ik}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^k} - m^2c^2 = 0. \tag{V.6}$$

B. Eikonal equation

The equation for the geodesic obtained in Lecture 4 is not applicable to the propagation of light since $ds = 0$. However, we can introduce some scalar parameterλ varying along world line of the light signal and then introduce a vector

$$k^i = \frac{dx^i}{d\lambda}, \tag{V.7}$$

which is tangent to the word line. This vector is called the four- dimensional wave vector. In the absence of a gravitational field according to the geometrical optics the propagation of light is given by the equation

$$dk^i = 0. \tag{V.8}$$

We know that the generalization of this equation in General Relativity is straightforward: $d \to D$. Then from $Dk^i = 0$ we obtain

$$\frac{dk^i}{d\lambda} + \Gamma^i_{kl} k^k k^l = 0. \tag{V.9}$$

From the definition of the four-vector for light (V.7) we have

$$ds^2 = g_{ik} dx^i dx^k = g_{ik} k^i k^k d\lambda^2, \tag{V.10}$$

then taking into account that $ds = 0$, we obtain

$$k_i k^i = g^{ik} k_i k_k = 0. \tag{V.11}$$

We know that any covariant vector can be presented as the gradient of a scalar

$$k_i = -\frac{\partial \Psi}{\partial x^i}, \tag{V.12}$$

were Ψ is a scalar. And we immediately obtain the Eikonal Equation in gravitational field

$$g^{ik} \frac{\partial \Psi}{\partial x^i} \frac{\partial \Psi}{\partial x^k} = 0. \tag{V.13}$$

The physical meaning of Ψ, which is called the Eikonal, follows from the obvious relationship

$$\Psi = -\int k_i dx^i, \tag{V.14}$$

which looks like the phase of the electromagnetic wave. We can see that the General Relativity can easily solve the problem of propagation of electromagnetic signals in the presence of a gravitational field, while the Newtonian gravity can not even offer more or less self consistent approach to the problem.
The shortest way to obtain the Eikonal equation is just to put $m = 0$ in the HamiltonJacobi equation and change notations.

C. The motion in a spherically symmetric static gravitational field

As an example of the motion of a test particle in a given gravitational field, let us consider a spherically symmetric gravitational field and assume that this field does not depend on time, i.e. it is static field. Taking into account the spherical symmetry we can choose our spherical coordinates in a such way that the plane of orbit coincides with the equatorial plane $\theta = \pi/2$ and $d\theta = 0$. Obviously, all the components of a metric tensor are functions of the radial coordinate only. Let us denote the radial coordinate as $x^1 = r$.
We can write the interval describing such gravitational field as

$$ds^2 = g_{00}(r) c^2 dt^2 + g_{11}(r) dr^2 + g_{33} d\phi^2. \tag{V.15}$$

In this case the Hamilton−Jacobi equation can be written as

$$g^{00}(r)\left(\frac{\partial S}{c\partial t}\right)^2 + g^{11}(r)\left(\frac{\partial S}{\partial r}\right)^2 + g^{33}(r)\left(\frac{\partial S}{\partial \phi}\right)^2 - m^2c^2 = 0. \tag{V.16}$$

Since all coefficients in this equation do not depend on t and ϕ we can say that

$$\frac{\partial S}{\partial t} = -E, \text{ and } \frac{\partial S}{\partial \phi} = L, \tag{V.17}$$

where E and L are constants, which by definition are the energy and angular momentum of the particle under consideration. Then putting

$$S = -Et + L\phi + S_r(r) \tag{V.18}$$

into the Hamilton−Jacobi equation we have

$$g^{00}(r)\frac{E^2}{c^2} + g^{11}(r)\left(\frac{dS_r(r)}{dr}\right)^2 + g^{33}(r)L^2 - m^2c^2 = 0, \tag{V.19}$$

hence

$$g^{11}(r)\left(\frac{dS_r(r)}{dr}\right)^2 = -g^{00}(r)\frac{E^2}{c^2} - g^{33}(r)L^2 + m^2c^2, \tag{V.20}$$

and

$$\frac{dS_r(r)}{dr} = \pm\sqrt{-\frac{1}{g^{11}(r)}\left(g^{00}(r)\frac{E^2}{c^2} + g^{33}(r)L^2 - m^2c^2\right)} = \pm mc\sqrt{-\frac{g^{00}(r)}{g^{11}(r)}\left(\tilde{E}^2 + \frac{g^{33}(r)}{g^{00}(r)}\tilde{L}^2 - \frac{1}{g^{00}(r)}\right)}, \tag{V.21}$$

where

$$\tilde{E} = \frac{E}{mc^2} \text{ and } \tilde{L} = \frac{L}{mc}. \tag{V.22}$$

Then we can calculate the radial component of the 4-velocity:

$$u^1 \equiv \frac{dr}{ds} = \frac{p^1}{mc} = g^{11}(r)\frac{p_1}{mc} = -g^{11}(r)\frac{\partial S}{mc\partial r} = -g^{11}(r)\frac{dS_r(r)}{mcdr} = \mp\sqrt{-g^{00}(r)g^{11}(r)\left(\tilde{E}^2 - U^2(r)_{eff}\right)}, \tag{V.23}$$

where

$$U^2_{eff}(r) = \frac{1}{g^{00}(r)}\left(1 - g^{33}(r)\tilde{L}^2\right) \tag{V.24}$$

is the so called "effective" potential. One can see that the condition

$$\frac{E}{mc^2} > U_{eff} \tag{V.25}$$

determines the admissible range of the motion. The effective potential includes in the relativistic manner potential energy plus kinetic energy of non-radial motion, this kinetic energy is determined by angular momentum L.
The radius of stable and unstable circular orbits is obtained from the simultaneous solution of the equations

$$U_{eff} = \tilde{E} \ \textit{and} \ \frac{dU_{eff}}{dr} = 0. \tag{V.26}$$

Back to Content
Previous Lecture
Next Lecture